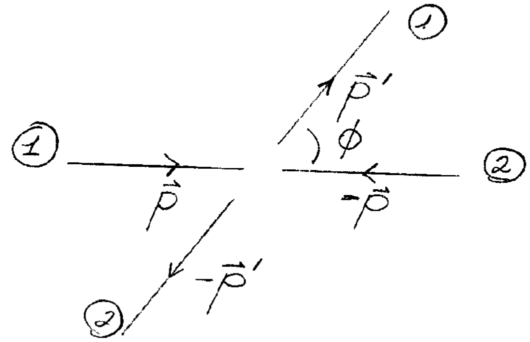
HOMEWORK 11 - PROBLEM 32



VIEW OF THE COLLISION FROM THE CENTER OF MOMENTUM (COM) FRAME.

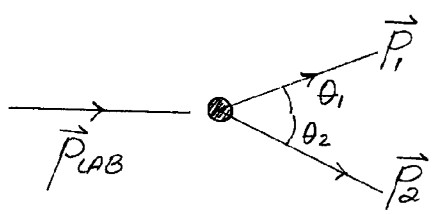
IN THIS FRAME THE FOUR MOMENTUM OF THE PARTICLES BEFORE AND AFTER THE COLLISION ARE GIVEN BY:

$$p_1^{\mu} = (E, p, 0, 0) \quad ; \quad p_1'^{\mu} = (E, p\cos\phi, p\sin\phi, 0)$$
$$p_2^{\mu} = (E, -p, 0, 0) \qquad\quad p_2'^{\mu} = (E, -p\cos\phi, -p\sin\phi, 0)$$

WHERE ENERGY-MOMENTUM CONSERVATION DEMANDS:

$$|\vec{p}\,'| = |\vec{p}| \equiv p = m\gamma\beta \quad \text{AND} \quad E = m\gamma \qquad (\text{I AM USING } c \equiv 1)$$

THE SAME COLLISION IN THE LABORATORY FRAME LOOKS LIKE:



VIEW OF THE COLLISION FROM THE LABORATORY (LAB) FRAME.

TO TRANSFORM THE FOUR MOMENTA FROM THE "COM" FRAME TO THE "LAB" FRAME WE JUST USE THE FOLLOWING LORENTZ TRANSFORMATION

$$A = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CLEARLY, UNDER THIS TRANSFORMATION p_2^μ (THE FOUR MOMENTUM OF THE TARGET PARTICLE) GOES TO:

$$p_2^\mu\big|_{LAB} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ -p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma E - \beta\gamma p \\ \beta\gamma E - \gamma p \\ 0 \\ 0 \end{pmatrix}$$

BUT...

$$\gamma E - \beta\gamma p = \gamma m \gamma - \beta\gamma m \gamma \beta = m\gamma^2(1-\beta^2) = m$$
$$\beta\gamma E - \gamma p = \beta\gamma^2 m - \gamma m \gamma \beta = 0$$

HENCE, AS EXPECTED UNDER THIS LORENTZ TRANSFORMATION $p_2^\mu\big|_{LAB} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

WE NOW TRANSFORM $p_1'^\mu$ AND $p_2'^\mu$; THAT IS,

$$p_1'^\mu\big|_{LAB} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p\cos\phi \\ p\sin\phi \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma E + \beta\gamma p\cos\phi \\ \beta\gamma E + \gamma p\cos\phi \\ p\sin\phi \\ 0 \end{pmatrix}$$

AND

$$p_2'^\mu\big|_{LAB} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ -p\cos\phi \\ -p\sin\phi \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma E - \beta\gamma p\cos\phi \\ \beta\gamma E - \gamma p\cos\phi \\ -p\sin\phi \\ 0 \end{pmatrix}$$

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IN THIS WAY,

$$\tan\theta_1 = \left.\frac{(P_1')_y}{(P_1')_x}\right|_{LAB} = \frac{p\sin\phi}{\beta\gamma E+\gamma p\cos\phi} = \frac{m\gamma\beta\sin\phi}{m\gamma^2\beta+m\gamma^2\beta\cos\phi} = \frac{\sin\phi}{\gamma(1+\cos\phi)}$$

SIMILARLY, (DEFINING θ_2 AS IN THE FIGURE)

$$\tan\theta_2 = \left.\frac{(P_2')_y}{(P_2')_x}\right|_{LAB} = \frac{p\sin\phi}{\beta\gamma E-\gamma p\cos\phi} = \frac{m\gamma\beta\sin\phi}{m\gamma^2\beta-m\gamma^2\beta\cos\phi} = \frac{\sin\phi}{\gamma(1-\cos\phi)}$$

SINCE WE HAVE ALREADY COMPUTED THE TANGENT OF BOTH ANGLES LET'S COMPUTE THEIR SUM THROUGH:

$$\tan(\theta) = \tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

$$= \frac{\dfrac{\sin\phi}{\gamma(1+\cos\phi)} + \dfrac{\sin\phi}{\gamma(1-\cos\phi)}}{1-\dfrac{\sin^2\phi}{\gamma^2(1-\cos^2\phi)}} = \frac{\dfrac{2\sin\phi}{\gamma\sin^2\phi}}{\dfrac{\gamma^2-1}{\gamma^2}}$$

OR USING,

$$\gamma^2-1 = \frac{1}{1-\beta^2} - 1 = \beta^2\gamma^2$$

WE OBTAIN

$$\boxed{\tan\theta = \frac{2}{\gamma\beta^2\sin\phi}}$$

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SOME NUMBERS...

β	ϕ	θ_1	θ_2	θ
0.01	15	7.50	82.50	90.00
	30	15.00	75.00	90.00
	60	30.00	60.00	90.00
	90	45.00	45.00	90.00
0.99	15	1.06	46.98	48.04
	30	2.17	27.77	29.93
	60	4.66	13.73	18.39
	90	8.03	8.03	16.06

HOMEWORK 11 – PROBLEM 33

LET US RECALL THAT THE FOUR-VECTORS A^μ AND ∂^μ ARE GIVEN BY:

$$A^\mu = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} \quad \text{AND} \quad \partial^\mu = \begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\vec{\nabla} \end{pmatrix}$$

IN THIS WAY WE OBTAIN:

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\frac{1}{c}\frac{\partial A^i}{\partial t} - (\nabla\phi)^i = E^i$$

AND

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\frac{\partial A^j}{\partial x^i} + \frac{\partial A^i}{\partial x^j} = -\epsilon_{ijk} B^k$$

SO THAT

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

THE TENSOR <u>DUAL</u> TO $F_{\mu\nu}$ IS DEFINED AS,

$$G^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

THEN,

$$G^{0i} = \frac{1}{2}\epsilon^{0i\alpha\beta} F_{\alpha\beta} = \frac{1}{2}\epsilon^{0ijk} F_{jk} = \frac{1}{2}\epsilon_{ijk}\left(-\epsilon_{jkl} B^l\right) = -B_i$$

AND

$$G^{ij} = \frac{1}{2}\epsilon^{ij0k} F_{0k} + \frac{1}{2}\epsilon^{ijk0} F_{k0} = \epsilon^{0ijk} F_{0k} = +\epsilon_{ijk} E_k$$

OR

$$\boxed{G^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}}$$

HOMEWORK II – PROBLEM 34

a)

WE HAVE COMPUTED IN PROBLEM 34:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}; \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

AND

$$G_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}; \quad G^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

HENCE,

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= -F^{\mu\nu}F_{\nu\mu} = -Tr\left(F^{\uparrow}F_{\downarrow}\right) = \\ &= -\left(E^2+E_x^2-B_z^2-B_y^2+E_y^2-B_z^2-B_x^2+E_z^2-B_y^2-B_x^2\right) \\ &= 2\left(B^2-E^2\right) \end{aligned}$$

AND

$$\begin{aligned} G^{\mu\nu}G_{\mu\nu} &= -G^{\mu\nu}G_{\nu\mu} = -Tr\left(G^{\uparrow}G_{\downarrow}\right) = \\ &= -\left(B^2+B_x^2-E_z^2-E_y^2+B_y^2-E_z^2-E_x^2+B_z^2-E_y^2-E_x^2\right) \\ &= 2\left(E^2-B^2\right) \end{aligned}$$

AND

$$\begin{aligned} F^{\mu\nu}G_{\mu\nu} &= -F^{\mu\nu}G_{\nu\mu} = -Tr\left(F^{\uparrow}G_{\downarrow}\right) = \\ &= -\left(\vec{E}\cdot\vec{B}+\vec{E}\cdot\vec{B}+\vec{E}\cdot\vec{B}+\vec{E}\cdot\vec{B}\right) \\ &= -4\vec{E}\cdot\vec{B} \end{aligned}$$

OR

$$\boxed{\begin{aligned} F^{\mu\nu}F_{\mu\nu} &= 2\left(B^2-E^2\right) = -G^{\mu\nu}G_{\mu\nu} \\ F^{\mu\nu}G_{\mu\nu} &= -4\vec{E}\cdot\vec{B} \end{aligned}}$$

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b) IMPOSSIBLE! WE HAVE JUST SHOWN THAT E^2-B^2 IS A LORENTZ INVARIANT. THUS, IF IN THE FIRST FRAME $\vec{E}\neq 0$ AND $\vec{B}=0$, SO THAT $\vec{E}^2-\vec{B}^2>0$; THEN, $\vec{E}^2-\vec{B}^2$ MUST BE GREATER THAN ZERO IN <u>ALL FRAMES.</u>