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Date: APR 2 0 2003

HOMEWORK 12 – PROBLEM 35

ASSUME THE FOLLOWING SOLUTION:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (A\cos\omega t + B\sin\omega t)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

THEN, THE COUPLED EQUATIONS OF MOTION REDUCE TO

$$-\omega^2\eta_1 - \left(\frac{m}{M}\right)\omega^2\eta_2 + \omega_0^2\eta_1 = 0$$

$$-\omega^2\eta_2 - \left(\frac{m}{M}\right)\omega^2\eta_1 + \omega_0^2\eta_2 = 0$$

OR

$$\boxed{\begin{pmatrix} \omega^2-\omega_0^2 & \left(\frac{m}{M}\right)\omega^2 \\ \left(\frac{m}{M}\right)\omega^2 & \omega^2-\omega_0^2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = 0}$$

IN ORDER FOR THIS SYSTEM OF EQUATIONS TO HAVE A NON-TRIVIAL SOLUTION WE DEMAND THAT THE DETERMINANT OF THE MATRIX BE EQUAL TO ZERO. THAT IS,

$$(\omega^2-\omega_0^2)^2 - \mu^2\omega^4 = 0\ ; \quad \text{WITH } \mu \equiv m/M$$

OR

$$(1-\mu^2)\omega^4 - 2\omega_0^2\omega^2 + \omega_0^4 = 0$$

OR

$$\boxed{\begin{aligned} \omega_1^2 &= \frac{\omega_0^2}{1+\mu} = \left(\frac{M}{M+m}\right)\omega_0^2 \\ \omega_2^2 &= \frac{\omega_0^2}{1-\mu} = \left(\frac{M}{M-m}\right)\omega_0^2 \end{aligned}}$$

THESE ARE THE EIGENVALUES

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NOW FOR THE EIGENVECTORS:

$$0 = (\omega_1^2 - \omega_0^2)\eta_1 + \mu\,\omega_1^2\,\eta_2$$
$$= -\left(\frac{\mu}{1+\mu}\right)\omega_0^2\,\eta_1 + \left(\frac{\mu}{1+\mu}\right)\omega_0^2\,\eta_2$$

OR $\eta_1 = \eta_2$

$$0 = (\omega_2^2 - \omega_0^2)\eta_1 + \mu\,\omega_2^2\,\eta_2$$
$$= \left(\frac{\mu}{1-\mu}\right)\omega_0^2\,\eta_1 + \left(\frac{\mu}{1-\mu}\right)\omega_0^2\,\eta_2$$

OR $\eta_1 = -\eta_2$

THUS,

$$\boxed{\begin{aligned} \omega_1^2 &= \left(\frac{M}{M+m}\right)\omega_0^2; \quad |\omega_1\rangle = \begin{pmatrix}1\\1\end{pmatrix} \\ \omega_2^2 &= \left(\frac{M}{M-m}\right)\omega_0^2; \quad |\omega_2\rangle = \begin{pmatrix}1\\-1\end{pmatrix} \end{aligned}}$$

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THE LAGRANGIAN FOR THE SYSTEM IS GIVEN BY

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2}\left[k_1\left(x_1^2 + x_3^2\right) + k_2 x_2^2 + k_3\left(x_1 x_2 + x_2 x_3\right)\right]$$

$$\equiv \frac{1}{2} \dot{\eta}^T \hat{T} \dot{\eta} - \frac{1}{2} \eta^T \hat{V} \eta \quad ; \quad \eta^T \equiv (x_1, x_2, x_3)$$

WHERE

$$\hat{T} = m\hat{t}\,; \quad \hat{t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} k_1 & k_3/2 & 0 \\ \frac{k_3}{2} & k_2 & k_3/2 \\ 0 & k_3/2 & k_1 \end{pmatrix} \equiv \frac{k_3}{2} \begin{pmatrix} \alpha_1 & 1 & 0 \\ 1 & \alpha_2 & 1 \\ 0 & 1 & \alpha_1 \end{pmatrix} \equiv \frac{k_3}{2} \hat{v}$$

WITH

$$\alpha_1 \equiv \frac{2k_1}{k_3} = \frac{2k_1}{\sqrt{2k_1 k_2}} = \sqrt{\frac{2k_1}{k_2}}$$

$$\alpha_2 \equiv \frac{2k_2}{k_3} = \frac{2k_2}{\sqrt{2k_1 k_2}} = \sqrt{\frac{2k_2}{k_1}}$$

THE EIGENVALUE EQUATION IS GIVEN BY:

$$0 = -\omega^2 \hat{T} |\eta\rangle + \hat{V} |\eta\rangle = -\omega^2 m |\eta\rangle + \frac{k_3}{2} \hat{v} |\eta\rangle$$

OR

$$\boxed{\hat{v} |\eta\rangle = \left(\frac{2\omega^2 m}{k_3}\right) |\eta\rangle \equiv \lambda |\eta\rangle}$$

EIGENVALUE EQUATION

$\lambda \equiv 2\omega^2 m / k_3$

WE MUST DIAGONALIZE THE MATRIX $\hat{v}$, THAT IS, WE MUST DEMAND THAT

$$\det(\hat{v} - \lambda \mathbb{1}) = \begin{vmatrix} \alpha_1 - \lambda & 1 & 0 \\ 1 & \alpha_2 - \lambda & 1 \\ 0 & 1 & \alpha_1 - \lambda \end{vmatrix} = 0$$

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OR

$$\det(\hat{v}-\lambda\mathbb{1}) = (\alpha_1-\lambda)\left[(\alpha_1-\lambda)(\alpha_2-\lambda)-1\right] - 1\left[\alpha_1-\lambda\right]$$
$$= (\alpha_1-\lambda)\left[(\alpha_1-\lambda)(\alpha_2-\lambda)-2\right] = 0$$

NOTE THAT,

$$(\alpha_1-\lambda)(\alpha_2-\lambda)-2 = \lambda^2 - (\alpha_1+\alpha_2)\lambda + \alpha_1\alpha_2 - 2$$

BUT $\alpha_1\alpha_2 = \left(\frac{2K_1}{K_2}\right)^{1/2}\left(\frac{2K_2}{K_1}\right)^{1/2} = 2 \Rightarrow \alpha_1\alpha_2 - 2 = 0$

SO THAT $(\alpha_1-\lambda)(\alpha_2-\lambda)-2 = \lambda\left(\lambda-(\alpha_1+\alpha_2)\right)$

OR

$$\det(\hat{v}-\lambda\hat{\mathbb{1}}) = \lambda(\alpha_1-\lambda)\left[\lambda-(\alpha_1+\alpha_2)\right] = 0$$

OR

$$\boxed{\lambda_1 = 0\ ;\ \lambda_2 = \alpha_1,\ \text{AND}\ \lambda_3 = \alpha_1+\alpha_2}$$

WITH $\omega^2 = \lambda K_3/2m$

EVEN THE EIGENVECTORS ARE NOT THAT BAD:

$$\boxed{|\lambda_1=0\rangle = \begin{pmatrix}1\\-\alpha_1\\1\end{pmatrix};\quad |\lambda_2=\alpha_1\rangle = \begin{pmatrix}1\\0\\-1\end{pmatrix};\quad |\lambda_3=\alpha_1+\alpha_2\rangle = \begin{pmatrix}\alpha_1/2\\1\\\alpha_1/2\end{pmatrix}}$$

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a) THE POSITION OF THE PENDULUM IS GIVEN BY:

$$x_p = x + \ell \sin\theta \simeq x + \ell\theta \equiv x + z$$
$$y_p = \ell\cos\theta \simeq \ell\left(1 - \frac{\theta^2}{2}\right) = \ell - \frac{z^2}{2\ell}$$

SO THE LAGRANGIAN OF THE SYSTEM IS GIVEN BY:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right) + mgy_p$$
$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}+\dot{z})^2 - mgz^2/2\ell + mg\ell + \text{HIGHER-ORDER}$$

THUS, IGNORING THE IRRELEVANT CONSTANT $mg\ell$ WE OBTAIN THE LAGRANGIAN OF SMALL OSCILLATIONS:

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2 + 2\dot{x}\dot{z}\right) - \frac{mg}{2\ell}z^2$$
$$= \frac{1}{2}(\dot{x},\dot{z})\begin{pmatrix} M+m & m \\ m & m \end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{z}\end{pmatrix} - \frac{1}{2}(x,z)\begin{pmatrix} 0 & 0 \\ 0 & mg/\ell \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}$$

OR

$$\boxed{L \equiv \frac{1}{2}\dot{q}^T\hat{T}\dot{q} - \frac{1}{2}q^T\hat{V}q}$$

WITH

$$\boxed{\hat{T} = m\begin{pmatrix} 1+\mu & 1 \\ 1 & 1 \end{pmatrix}; \quad \hat{V} = \frac{mg}{\ell}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

$\mu \equiv M/m$; LAGRANGIAN OF SMALL OSCILLATIONS.

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WE NEED TO SOLVE THE EIGENVALUE EQUATION:

$$0 = (-\omega^2 \hat{T} + \hat{V})|\eta\rangle$$
$$= \left(-\omega^2 m \hat{t} + \frac{mg}{\ell}\hat{v}\right)|\eta\rangle$$

OR

$$\hat{v}|\eta\rangle = \lambda \hat{t}|\eta\rangle \; ; \quad \lambda \equiv \omega^2/\omega_0^2 \; ; \; \omega_0^2 \equiv g/\ell$$

$$\hat{v} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ AND } \hat{t} = \begin{pmatrix} 1+\mu & 1 \\ 1 & 1 \end{pmatrix} \; ; \; \mu \equiv \frac{M}{m}$$

THUS, WE DEMAND THAT:

$$\det(\hat{v} - \lambda \hat{t}) = \begin{vmatrix} -\lambda(1+\mu) & -\lambda \\ -\lambda & 1-\lambda \end{vmatrix} = \lambda(\lambda-1)(1+\mu) - \lambda^2 = 0$$

OR

$$\lambda\left[(\lambda-1)(1+\mu) - \lambda\right] = \lambda\left[\mu\lambda - (1+\mu)\right] = 0$$

WHICH IMPLIES $\lambda_1 = 0$ AND $\lambda_2 = \frac{(1+\mu)}{\mu}$

FOR THE FIRST EIGENVECTOR WE HAVE $-\lambda_1 \eta_1 + (1-\lambda)\eta_2 = \eta_2 = 0$

WHILE FOR THE SECOND ONE $-\lambda_2 \eta_1 + (1-\lambda_2)\eta_2 = 0$

OR

$$\eta_2 = \left(\frac{\lambda_2}{1-\lambda_2}\right)\eta_1 = -\frac{(1+\mu)/\mu}{1/\mu}\eta_1 = -(1+\mu)\eta_1$$

THUS,

$$\boxed{\begin{aligned} &\omega_1^2 = 0 \; ; \quad |\omega_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\omega_2^2 = \left(\frac{1+\mu}{\mu}\right)\omega_0^2 \; ; \; |\omega_2\rangle = \begin{pmatrix} 1 \\ -(1+\mu) \end{pmatrix} \end{aligned}}$$

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THE MOST GENERAL SOLUTION IS GIVEN BY

$$\begin{pmatrix} X(t) \\ Z(t) \end{pmatrix} = (A_1 + B_1 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (A_2 \cos\omega_2 t + B_2 \sin\omega_2 t)\begin{pmatrix} 1 \\ -1-\mu \end{pmatrix}$$

NOW USING THE INITIAL CONDITIONS WE OBTAIN:

$$\begin{pmatrix} X(0) \\ Z(0) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + A_2 \begin{pmatrix} 1 \\ -1-\mu \end{pmatrix} = \begin{pmatrix} 0 \\ Z_0 \end{pmatrix} \Rightarrow A_1 = -A_2 = \frac{Z_0}{1+\mu}$$

$$\begin{pmatrix} \dot{X}(0) \\ \dot{Z}(0) \end{pmatrix} = B_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \omega_2 B_2 \begin{pmatrix} 1 \\ -1-\mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow B_1 = B_2 = 0$$

SO THAT,

$$\boxed{X(t) = \frac{Z_0}{1+\mu}\left[1 - \cos\omega_2 t\right]; \quad Z(t) = Z_0 \cos\omega_2 t}$$