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DATE JAN 07 2002

PHYSICS 4241/5227 – PROBLEM 2

a) THE LAGRANGIAN FOR THE SYSTEM IS GIVEN BY

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$$

THUS, THE EULER-LAGRANGE'S EQUATION OF MOTION BECOMES:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = -x - \ddot{x} \quad \text{OR} \quad \ddot{x} + x = 0$$

AND THE MOST GENERAL SOLUTION IS GIVEN BY $x(t) = A\sin t + B\cos t$. NOW USING THE BOUNDARY CONDITIONS WE CAN DETERMINE A AND B. THAT IS,

$$x(t=0) = B = 0$$
$$x(t=\tfrac{\pi}{2}) = A = 1$$

OR

$$\boxed{x(t) = \sin t}$$

FOR THIS EXACT PATH THE ACTION IS GIVEN BY:

$$S[x(t)] = \int_0^{\pi/2} dt \left[\frac{1}{2}\cos^2 t - \frac{1}{2}\sin^2 t\right] = \frac{1}{2}\int_0^{\pi/2} \cos 2t \, dt = 0$$

THUS, FOR THE EXACT PATH:

$$\boxed{x(t) = \sin t \quad \text{AND} \quad S[x(t)] = 0}$$

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b) FOR A LINEAR PATH, $x(t) = a + bt$, SUBJECT TO THE BOUNDARY CONDITIONS:

$$x(0) = a = 0$$
$$x(\pi/2) = b\frac{\pi}{2} = 1 \Rightarrow b = \frac{2}{\pi}$$

THE APPROXIMATE PATH BECOMES: $x(t) = \frac{2}{\pi}t$ AND THE ACTION FOR THIS PATH,

$$S[x(t)] = \int_0^{\pi/2} dt \left[\frac{1}{2}\left(\frac{2}{\pi}\right)^2 - \frac{1}{2}\left(\frac{2}{\pi}\right)^2 t^2\right]$$
$$= \frac{2}{\pi^2}\int_0^{\pi/2} dt\,(1 - t^2) = \frac{2}{\pi^2}\left[\frac{\pi}{2} - \frac{1}{3}\left(\frac{\pi}{2}\right)^3\right]$$

OR

$$S[x(t)] = \frac{1}{\pi}\left[1 - \frac{\pi^2}{12}\right] = \frac{1}{\pi} - \frac{\pi}{12} \simeq 0.0565$$

HENCE, FOR THE LINEAR PATH WE OBTAIN:

$$\boxed{x(t) = \frac{2}{\pi}t \quad \text{AND} \quad S[x(t)] = \frac{1}{\pi} - \frac{\pi}{12} \simeq 0.0565 > 0}$$

c) FOR A QUADRATIC PATH, $x(t) = a + bt + ct^2$, THE BOUNDARY CONDITIONS DO NOT FIX ALL THE CONSTANTS. THAT IS,

$$x(0) = a = 0\ ; \quad x\left(\frac{\pi}{2}\right) = b\frac{\pi}{2} + c\left(\frac{\pi}{2}\right)^2 = 1$$

OR

$$b = \frac{2}{\pi}\left(1 - \frac{\pi^2}{4}c\right) = \frac{2}{\pi} - \frac{\pi}{2}c$$

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THUS, THE QUADRATIC PATH BECOMES: $x(t) = \left(\frac{2}{\pi} - \frac{\pi}{2}c\right)t + ct^2$

AND THE ACTION FOR THIS PATH:

$$S(c) = \int_0^{\pi/2} \frac{1}{2}\left(\dot{x}^2 - x^2\right) = \alpha c^2 + \beta c + \gamma, \text{ WHERE}$$

$$\alpha \equiv (40\pi^3 - \pi^5)/1920; \quad \beta = \pi^3/96; \quad \gamma = \frac{1}{\pi} - \frac{\pi}{12}$$

WE FIX THE SOLE REMAINING CONSTANT (i.e, c) BY DEMANDING S(c) TO BE MINIMIZED WITH RESPECT TO c. THAT IS,

$$S'(c) = 2\alpha c + \beta = 0 \rightarrow c_0 = -\beta/2\alpha = \frac{10}{\pi^2 - 40} \simeq -0.3319$$

SUBSTITUTING THIS VALUE OF c WE OBTAIN FOR THE QUADRATIC PATH:

$$\boxed{x(t) = \frac{(80 + 3\pi^2 - 10\pi t)t}{(40 - \pi^2)\pi}}$$

AND

$$\boxed{S[x(t)] = \frac{(3\pi^4 - 416\pi^2 + 3840)}{96(40 - \pi^2)\pi} \simeq 0.0029}$$

Problem 2:

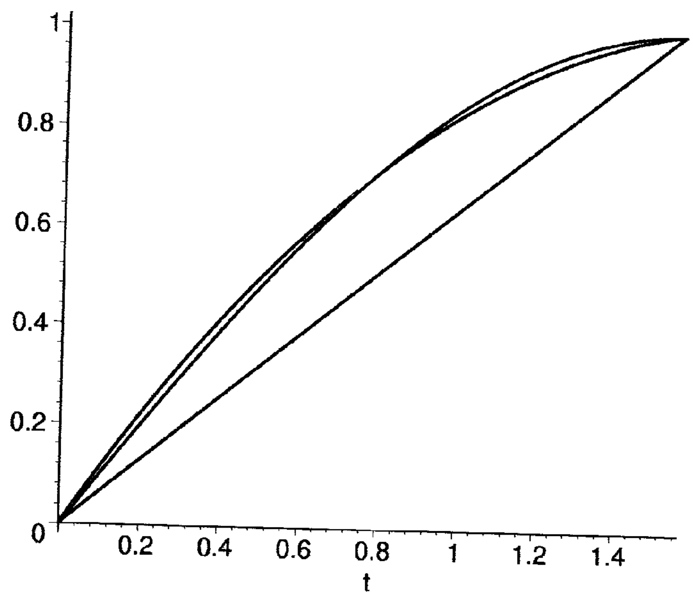
```
> restart:
> x0 := sin(t): y0 := 2*t/Pi: z0 := (80+3*Pi^2-10*Pi*t)*t/((40-Pi^2)*Pi):
> x1 := diff(x0,t): y1 := diff(y0,t): z1 := diff(z0,t):
> Le := (x1^2-x0^2)/2: L1 := (y1^2-y0^2)/2: Lq := (z1^2-z0^2)/2:
> Se := int(Le,t=0..Pi/2): S1 := int(L1,t=0..Pi/2): Sq := int(Lq,t=0..Pi/2):
> Se,S1,simplify(Sq);
```

$$0, \frac{1}{\pi} - \frac{1}{12}\pi, -\frac{1}{96}\frac{3\pi^4 - 416\pi^2 + 3840}{\pi(-40+\pi^2)}$$

```
> evalf(Se), evalf(S1), evalf(Sq);
```

0., .0565104983, .002913119

```
> plot([x0,y0,z0],t=0..Pi/2,thickness=3,color=[black,red,blue]);
```



PHYSICS 4241/5227 - PROBLEM 3

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a) THE LAGRANGIAN FOR THE SYSTEM IS GIVEN BY:

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) - Mgy$$

THE EQUATIONS OF MOTION ARE GIVEN BY:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 = M\ddot{x} \Rightarrow x(t) = a + bt$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 = M\ddot{y} + Mg \Rightarrow y(t) = c + dt - \frac{gt^2}{2}$$

AND BECOME, USING THE BOUNDARY CONDITIONS:

$$\boxed{x(t) = \frac{L}{T}t \; ; \quad y(t) = \frac{g}{2}t(T-t)}$$

IN PARTICULAR,

$$\dot{x}(0) = v_0 \cos\theta_0 = L/T$$
$$\dot{y}(0) = v_0 \sin\theta_0 = gT/2$$

OR

$$\boxed{v_0 = \left[(L/T)^2 + (gT/2)^2\right]^{1/2} ; \quad \tan\theta_0 = \frac{gT^2}{2L}}$$

b) THE ACTION FOR THIS EXACT TRAJECTORY IS GIVEN BY:

$$S[x(t), y(t)] = \int_0^T dt \left[\frac{1}{2}M\left(\frac{L}{T}\right)^2 + \frac{1}{2}M\left(\frac{g}{2}T - gt\right)^2 - Mg^2\frac{t}{2}(T-t)\right]$$

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DATE	

OR

$$\boxed{S[x(t),y(t)] = \frac{M}{24T}\left[12L^2 - g^2T^4\right]}$$

c) USE AN APPROXIMATE SINUSOIDAL PATH OF THE FORM:

$$y(t) = A\sin(\pi t/T)$$
$$\dot{y}(t) = \frac{\pi}{T}A\cos(\pi t/T)$$

TO COMPUTE THE ACTION FOR THIS PATH. THAT IS,

$$S_A[x(t),y(t)] = \int_0^T dt\left[\frac{1}{2}M\left(\frac{L}{T}\right)^2 + \frac{1}{2}M\left(\frac{\pi}{T}\right)^2A^2\cos^2(\pi t/T) - MgA\sin(\pi t/T)\right]$$
$$= \frac{M}{4\pi T}\left[\pi^3A^2 - 8gT^2A + 2\pi L^2\right]$$

MINIMIZING THE ACTION WITH RESPECT TO A WE OBTAIN: $A = 4gT^2/\pi^3$
AND THUS,

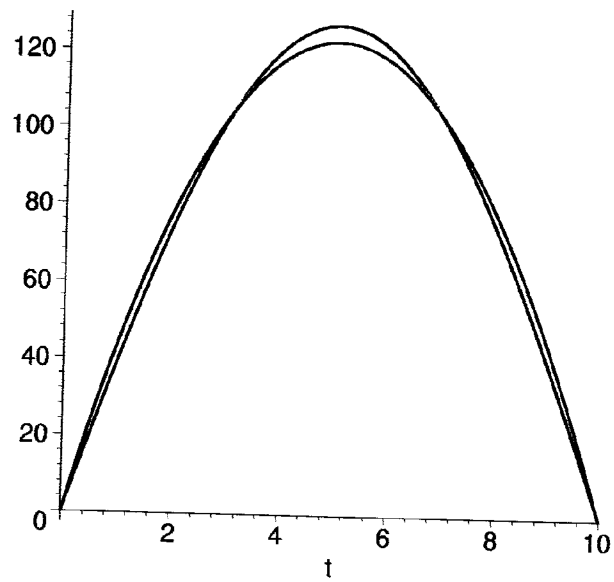
$$\boxed{\begin{aligned} y(t) &= \frac{4gT^2}{\pi^3}\sin(\pi t/T) \\ S[x(t),y(t)] &= \frac{M}{4\pi T}\left[-16\frac{g^2T^4}{\pi^3} + 2\pi L^2\right]\end{aligned}}$$

d) THE ACTION, OR RATHER THE ACTION IN UNITS OF M AND WITH THE CONTRIBUTION FROM THE X-COMPONENT OF THE KINETIC ENERGY REMOVED, YIELDS:

$$\boxed{S_{ex} = -4009.8375 < -3951.8323 = S_{aprox.}}$$

Problem 3:

```
restart:
> x0 := L*t/T: y0 := g*t*(T-t)/2: z0 := A*sin(Pi*t/T):
> x1 := diff(x0,t): y1 := diff(y0,t): z1 := diff(z0,t):
> Le := M*(x1^2+y1^2)/2-M*g*y0: Ls :=M*(x1^2+z1^2)/2-M*g*z0:
> Se := simplify(int(Le,t=0..T)): Ss0 := simplify(int(Ls,t=0..T)):
> Ss1 := diff(Ss0,A): A0 := solve(Ss1=0,A):
> z0 := subs(A=A0,z0): Ss := subs(A=A0,Ss0):
> yp := subs([g=9.81,T=10],y0): zp := subs([g=9.81,T=10],z0):
> plot([yp,zp],t=0..10,thickness=3,color=[black,red]);
```




```
> Sep := simplify(subs([g=9.81,T=10],Se)/M-L^2/20):
  Ssp := simplify(subs([g=9.81,T=10],Ss)/M-L^2/20):
> Sep; Ssp;
```

-4009.837500

-3951.832378