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Date: JAN 25 2003

HOMEWORK 3 – PROBLEM 7

a) IN SPHERICAL COORDINATES THE KINETIC ENERGY OF A PARTICLE OF MASS m IS GIVEN BY:

$$T=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\varphi}^2)$$

WHICH REDUCES IN THE CASE OF A SPHERICAL PENDULUM OF FIXED LENGTH $r=b=$ CONSTANT TO:

$$T=\frac{1}{2}mb^2(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2)$$

FOR THE POTENTIAL ENERGY WE HAVE:

$$V=mgz=mgb\cos\theta$$

NOTE THAT θ IS THE CONVENTIONALLY DEFINED POLAR ANGLE IN SPHERICAL COORDINATES; IT DIFFERS FROM THE ONE DISPLAYED IN FIG. 7-10 OF MARION AND THORNTON. OF COURSE, YOU CAN USE EITHER ONE AS LONG AS YOU ARE CAREFUL. THE LAGRANGIAN BECOMES:

$$\boxed{L=T-V=\frac{1}{2}mb^2(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2)-mgb\cos\theta}$$

b) THE TWO CONJUGATE MOMENTA ARE GIVEN BY

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=mb^2\dot{\theta} \quad \text{AND} \quad p_\varphi=mb^2\sin^2\theta\,\dot{\varphi}$$

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THEN,

$$h = \dot{\theta} p_\theta + \dot{\varphi} p_\varphi - L = \dot{\theta}(mb^2\dot{\theta}) + \dot{\varphi}(mb^2 \sin^2\theta\, \dot{\varphi}) - \left[\frac{1}{2} mb^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) - mgb\cos\theta\right]$$
$$= \frac{1}{2} mb^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) + mgb\cos\theta$$

OR

$$\boxed{h = T + V = E = \frac{1}{2} mb^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2) + mgb\cos\theta}$$

THE HAMILTONIAN IS OBTAINED FROM h BY USING

$$\dot{\theta} = \frac{p_\theta}{mb^2} \quad \text{AND} \quad \dot{\varphi} = \frac{p_\varphi}{mb^2 \sin^2\theta}$$

SO THAT

$$H = \frac{1}{2} mb^2 \left(\frac{p_\theta}{mb^2}\right)^2 + \frac{1}{2} mb^2 \sin^2\theta \left(\frac{p_\varphi}{mb^2 \sin^2\theta}\right)^2 + mgb\cos\theta$$

OR

$$\boxed{H(\theta, \varphi, p_\theta, p_\varphi) = \frac{p_\theta^2}{2mb^2} + \frac{p_\varphi^2}{2mb^2\sin^2\theta} + mgb\cos\theta}$$

c) TO OBTAIN THE EFFECTIVE POTENTIAL WE MUST FIRST REALIZE THAT φ IS A CYCLIC COORDINATE SO THAT p_φ IS A CONSTANT OF THE MOTION; INDEED, p_φ IS THE CONSERVED Z-COMPONENT OF THE ANGULAR MOMENTUM. SO WE CAN WRITE THE CONSERVED ENERGY OF THE SYSTEM AS:

$$E = \frac{p_\theta^2}{2mb^2} + V_{EFF}(\theta) = \frac{1}{2} mb^2 \dot{\theta}^2 + V_{EFF}(\theta)$$

$$V_{EFF}(\theta) \equiv \left(mgb\cos\theta + \frac{\ell^2}{2mb^2 \sin^2\theta}\right) \quad ; \quad \text{WITH } \ell \equiv p_\varphi = \text{CONSTANT}$$

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TO MAKE THINGS LOOK SIMPLER INTRODUCE THE FOLLOWING DEFINITIONS:

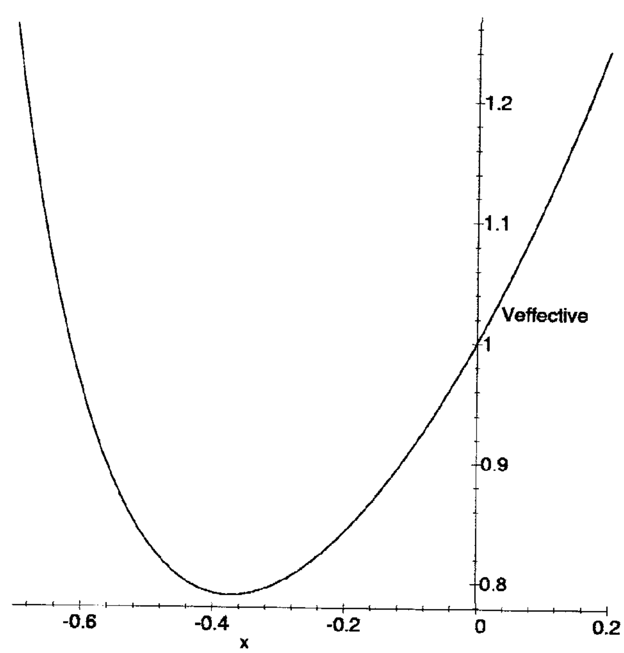
$$X \equiv \cos\theta \; ; \quad \alpha \equiv mgb \; ; \quad \beta \equiv \frac{\ell^2}{2mb^2}$$

THEN,

$$\boxed{V_{EFF}(x) = \alpha x + \frac{\beta}{1 - x^2}}$$

FOR THE PENDULUM TO OSCILLATE AT CONSTANT HEIGHT $(\theta = \theta_0)$ TWO CONDITIONS MUST BE FULLFILLED:

a) $V'_{EFF}(x_0) = 0$ AND b) $E = V_{EFF}(x_0)$



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THE FIRST CONDITION IMPLIES THAT

$$V'_{EFF}(\cdot) = \alpha + \frac{2x\beta}{(1-x^2)^2} = 0$$

OR

$$\boxed{x^4 - 2x^2 + \left(\frac{2\beta}{\alpha}\right)x + 1 = 0}$$

THERE IS NO ALGORITHM TO SOLVE THIS EQUATION SO IT MUST BE DONE NUMERICALLY. FOR EXAMPLE, FOR $\gamma=1$ THE SOLUTION IS $x_0 = -0.3715$ OR $\theta_0 = 111.809$ degrees.

ONCE x_0 IS FOUND, ONE MUST TUNE THE ENERGY SO THAT $E = V(x_0)$.

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HOMEWORK 3 - PROBLEM 8

a) THE TOTAL DERIVATIVE OF $g = g(q_k, p_k; t)$ IS GIVEN BY

$$\frac{dg}{dt} = \sum_k \left(\frac{\partial g}{\partial q_k} \dot{q}_k + \frac{\partial g}{\partial p_k} \dot{p}_k \right) + \frac{\partial g}{\partial t}$$

NOW WE USE HAMILTON'S EQUATIONS OF MOTION TO WRITE:

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{AND} \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

IN THIS WAY THE TOTAL DERIVATIVE OF g WITH RESPECT TO TIME BECOMES,

$$\frac{dg}{dt} = \sum_k \left(\frac{\partial g}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial g}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \frac{\partial g}{\partial t}$$

OR

$$\boxed{\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t}}$$

b) TRIVIALLY FROM part a) WE OBTAIN, (using $g = q_j$ AND $g = p_j$)

$$\boxed{\begin{aligned} \frac{dq_j}{dt} &= \dot{q}_j = [q_j, H] \\ \frac{dp_j}{dt} &= \dot{p}_j = [p_j, H] \end{aligned}}$$

HOMEWORK 3 – PROBLEM 9

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a)

$$[fg, h] = \frac{\partial}{\partial q}(fg)\frac{\partial h}{\partial p} - \frac{\partial h}{\partial q}\frac{\partial}{\partial p}(fg)$$

$$= \left\{\left(\frac{\partial f}{\partial q}\right)\left(\frac{\partial h}{\partial p}\right) - \left(\frac{\partial h}{\partial q}\right)\left(\frac{\partial f}{\partial p}\right)\right\} g$$

$$+ f\left\{\left(\frac{\partial g}{\partial q}\right)\left(\frac{\partial h}{\partial p}\right) - \left(\frac{\partial h}{\partial q}\right)\left(\frac{\partial g}{\partial p}\right)\right\} = [f,h]g + f[g,h]$$

FOR THE NEXT ONE WE COULD DO THE SAME OR SIMPLY USE THE FACT THAT:

$$[f, gh] = -[gh, f] = -g[h,f] - [g,f]h$$

$$= [f,g]h + g[f,h]$$

b)

$$[f(r), p_i] = \frac{\partial f}{\partial x_a}\frac{\partial p_i}{\partial p_a} - \overset{0}{\cancel{\frac{\partial p_i}{\partial x_a}}}\,\overset{0}{\cancel{\frac{\partial f}{\partial p_a}}} = \frac{\partial f}{\partial x_a}\delta_{ia}$$

$$= \frac{\partial f}{\partial x_i} = f'(r)\left(\frac{\partial r}{\partial x_i}\right) = \left(\frac{x_i}{r}\right) f'(r)$$

OR

$$\boxed{[f(r), p_i] = \frac{x_i}{r} f'(r)}$$

c)

i) $[x_i, L_j] = \epsilon_{jkl}[x_i, x_k p_l] = \epsilon_{jkl} x_k \overset{\delta_{il}}{[x_i, p_l]}$
$= \epsilon_{ijk} x_k$

ii) $[p_i, L_j] = \epsilon_{jkl}[p_i, x_k p_l] = \epsilon_{jkl} \overset{-\delta_{ik}}{[p_i, x_k]} p_l$
$= \epsilon_{ijl} p_l$

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iii) $[L_i, L_j] = \epsilon_{ikl} [x_k p_l, L_j]$

$$= \epsilon_{ikl} x_k [p_l, L_j] + \epsilon_{ikl} [x_k, L_j] p_l$$

$$= \epsilon_{ikl} \epsilon_{ljm} x_k p_m + \epsilon_{ikl} \epsilon_{kjm} x_m p_l$$

$$= (\delta_{ij}\delta_{km} - \delta_{im}\delta_{jk}) x_k p_m - (\delta_{ij}\delta_{lm} - \delta_{im}\delta_{jl}) x_m p_l$$

$$= (\vec{r}\cdot\vec{p})\delta_{ij} - x_j p_i - (\vec{r}\cdot\vec{p})\delta_{ij} + x_i p_j$$

$$= (x_i p_j - x_j p_i) = \epsilon_{ijk} L_k$$

HENCE,

$$\boxed{\begin{aligned} [x_i, L_j] &= \epsilon_{ijk} x_k \\ [p_i, L_j] &= \epsilon_{ijk} p_k \\ [L_i, L_j] &= \epsilon_{ijk} L_k \end{aligned}}$$