HOMEWORK 5 - PROBLEM 13

a) WE SHOW THE VALIDITY OF THE RELATION FOR EACH COMPONENT. THAT IS,

$$[\nabla\times(\nabla\phi(\vec{x}))]_i = \varepsilon_{ijk}\,\partial_j\partial_k\,\phi(\vec{x}) = 0$$

THE ABOVE RELATION VANISHES BECAUSE ε_{ijk} IS ANTISYMMETRIC UNDER THE EXCHANGE OF j AND k BUT $\partial_j\partial_k \equiv \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}$ IS SYMMETRIC.

b) THIS IS A SCALAR QUANTITY; THUS,

$$\nabla\cdot(\nabla\times\vec{A}) = \partial_i(\nabla\times\vec{A})_i = \partial_i\,\varepsilon_{ijk}\,\partial_j A_k = \varepsilon_{ijk}\,\partial_i\partial_j A_k = 0$$

c) THIS IS A VECTOR QUANTITY; AGAIN, WE SHOW THE VALIDITY OF THE RELATION FOR EACH COMPONENT. THAT IS,

$$[\nabla\times(\nabla\times\vec{A})]_i = \varepsilon_{ijk}\,\partial_j(\nabla\times\vec{A})_k = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,\partial_l A_m$$

$$= \varepsilon_{kij}\,\varepsilon_{klm}\,\partial_j\partial_l A_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\partial_l A_m$$

$$= \partial_i(\partial_j A_j) - (\partial_j\partial_j)A_i = \partial_i(\nabla\cdot\vec{A}) - \nabla^2 A_i$$

OR

$$\boxed{\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}}$$

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FEB 09 2003

Homework 5 - Problem 14

a) Let $\vec{B} = \nabla\times\vec{A}_1$ and $\vec{A}_2 = \vec{A}_1 + \nabla\Lambda$, then

$$\nabla\times\vec{A}_2 = \nabla\times\vec{A}_1 + \underbrace{\nabla\times(\nabla\Lambda)}_{0} = \vec{B}$$

Hence,

$$\boxed{\begin{array}{l}\vec{B} = \nabla\times\vec{A}_1 = \nabla\times\vec{A}_2 \\ \vec{A}_1, \vec{A}_2 \text{ ARE GAUGE EQUIVALENT VECTOR POTENTIALS}\end{array}}$$

b) I start with $\vec{A}_2(\vec{x}) = Bx\hat{y}$. Then,

$$(\nabla\times\vec{A}_2)_i = \epsilon_{ijk}\partial_j A_{2k} = \epsilon_{ijk}\partial_j [xB]\delta_{k2} = \epsilon_{ijk} B\delta_{j1}\delta_{k2}$$
$$= B\epsilon_{i12} = B\delta_{i3}$$

Moreover,

$$\nabla\cdot\vec{A}_2 = \partial_i A_{2i} = \partial_i (xB)\delta_{i2} = \partial_2(xB) \equiv 0$$

Hence,

$$\boxed{\nabla\times\vec{A}_2 = B\hat{z} \text{ AND } \nabla\cdot\vec{A}_2 = 0}$$

Similarly for $\vec{A}_3(\vec{x}) = -By\hat{x}$:

$$(\nabla\times\vec{A}_3)_i = \epsilon_{ijk}\partial_j A_{3k} = \epsilon_{ijk}\partial_j(-yB)\delta_{k1} = -\epsilon_{ijk} B\delta_{j2}\delta_{k1}$$
$$= -B\epsilon_{i21} = +B\epsilon_{12i} = B\delta_{i3}$$

And

$$\nabla\cdot\vec{A}_3 = \partial_i A_{3i} = \partial_i(-yB)\delta_{i1} = \partial_1(-yB) = 0$$

Thus,

$$\boxed{\nabla\times\vec{A}_3 = B\hat{z} \text{ AND } \nabla\cdot\vec{A}_3 = 0}$$

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FINALLY $\vec{A}_1(\vec{x}) = \frac{B}{2}(x\hat{y} - y\hat{x}) = \frac{1}{2}(\vec{A}_2 + \vec{A}_3)$. THEN

AND

$$\boxed{\begin{aligned} \nabla\times\vec{A}_1 &= \frac{1}{2}\left(\nabla\times\vec{A}_2 + \nabla\times\vec{A}_3\right) = B\hat{z} \\ \nabla\cdot\vec{A}_1 &= \frac{1}{2}\left(\nabla\cdot\vec{A}_2 + \nabla\cdot\vec{A}_3\right) = 0 \end{aligned}}$$

c) WE DEFINE: $\vec{A}_2(\vec{x}) \equiv \vec{A}_1(\vec{x}) + \nabla\Lambda_{12}(\vec{x})$. THEN, LOOKING AT THIS RELATION COMPONENT BY COMPONENT WE OBTAIN:

$$0 = -\frac{B}{2}y + \frac{\partial\Lambda_{12}}{\partial x} \Rightarrow \frac{\partial\Lambda_{12}}{\partial x} = \frac{B}{2}y$$

$$Bx = \frac{B}{2}x + \frac{\partial\Lambda_{12}}{\partial y} \Rightarrow \frac{\partial\Lambda_{12}}{\partial y} = \frac{B}{2}x$$

CLEARLY, BOTH RELATIONS ARE SATISFIED IF $\Lambda_{12}(\vec{x}) = \frac{B}{2}xy$.

NOW DEFINE $\vec{A}_3(\vec{x}) \equiv \vec{A}_2(\vec{x}) + \nabla\Lambda_{23}(\vec{x})$. IN THIS CASE WE OBTAIN:

$$-By = 0 + \frac{\partial\Lambda_{23}}{\partial x} \Rightarrow \frac{\partial\Lambda_{23}}{\partial x} = -By$$

$$0 = +Bx + \frac{\partial\Lambda_{23}}{\partial y} \Rightarrow \frac{\partial\Lambda_{23}}{\partial y} = -Bx$$

SO THAT $\Lambda_{23}(\vec{x}) = -Bxy$

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FINALLY, $\vec{A}_1(\bar{x}) = \vec{A}_3(\bar{x}) + \nabla\Lambda_{31}(\bar{x})$. TO OBTAIN $\Lambda_{31}(\bar{x})$ WE SIMPLY ADD THE PREVIOUS TWO EQUATIONS. THAT IS,

$$A_2 + A_3 = A_1 + A_2 + \nabla(\Lambda_{12} + \Lambda_{23})$$

OR

$$\vec{A}_1(\bar{x}) = \vec{A}_3(\bar{x}) - \nabla(\Lambda_{12} + \Lambda_{23}) \Rightarrow \Lambda_{31}(\bar{x}) = \frac{B}{2}xy$$

THUS,

$$\boxed{\Lambda_{12}(\bar{x}) = \Lambda_{31}(\bar{x}) = -\frac{1}{2}\Lambda_{23}(\bar{x}) = \frac{B}{2}xy}$$

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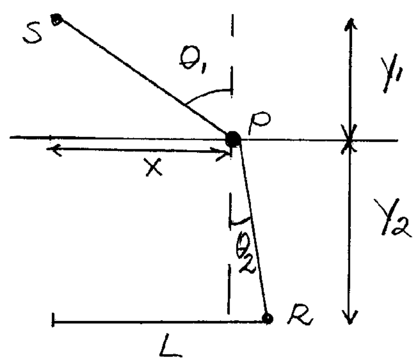
DATE FEB 04 2002

Homework 5 - Problem 15



The distance travelled from the source S to the point P in the interface is: $d_1 = (x^2 + y_1^2)^{1/2}$, while from point P to the receptor R is $d_2 = [(L-x)^2 + y_2^2]^{1/2}$. Hence, the total time travelled is given by:

$$t(x) = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{1}{v_1}(x^2 + y_1^2)^{1/2} + \frac{1}{v_2}\left[(L-x)^2 + y_2^2\right]^{1/2}$$

Thus, the path of least time is determined from

$$t'(x) = \frac{x}{v_1\sqrt{x^2 + y_1^2}} - \frac{(L-x)}{v_2\sqrt{(L-x)^2 + y_2^2}} = 0$$

OR

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \Rightarrow \boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2} \quad ; \; n \equiv c/v$$

Snell's Law