STATISTICAL MECHANICS

Final Exam – Spring 2003

PROBLEM #1 : (10 points) Use the grand canonical ensemble to find the expression of the entropy for a gas of non-interacting particles obeying (i) Bose-Einstein statistics, and (ii) Fermi-Dirac statistics in terms of the mean occupation number of a single-particle state with energy ϵ, $\langle n_\epsilon \rangle$.

Hint: In the grand canonical ensemble the entropy is given by

$$S = kT \left(\frac{\partial \ln(\mathcal{Z})}{\partial T} \right)_{z,V} - \langle N \rangle k \ln(z) + k \ln(\mathcal{Z})$$

where $\mathcal{Z}$ and $\langle N \rangle$ are the grand partition function and the expectation value of the total number of particles, respectively.

PROBLEM #2 : (12 points) Consider a two-dimensional ideal Bose gas. Let $V = L^2$ be the area available to the system. The number of particles (which is conserved) is given by

$$N = z\frac{\partial}{\partial z} \ln[\mathcal{Z}(z,V,T)] = \sum_{\mathbf{p}} \left[z^{-1} exp(\beta\epsilon_{\mathbf{p}}) - 1\right]^{-1} \quad ,$$

where $\mathcal{Z}$ is the grand partition function.

(a) Consider the system in the thermodynamic limit and discuss the importance of the $\mathbf{p} = 0$ state.

(b) Show that there is no Bose-Einstein condensation at $T \neq 0$ for a two-dimensional ideal gas.

(c) What is the groundstate of the system?

PROBLEM #3 : (8 points) The atom He^3 consists of three nucleons, has spin 1/2 and is a fermion. The density of liquid He^3 is $0.081 g/cm^3$ near absolute zero. Calculate the Fermi energy ϵ_F (in meV) and the Fermi temperature T_F (in kelvin).

$\hbar = 1.055 \times 10^{-27} erg/s$, $m_{nucleon} = 1.67 \times 10^{-24} g$, $k_B = 1.38 \times 10^{-16} erg K^{-1}$, $1eV = 1.6022 \times 10^{-12} erg$.

Final Exam - Solutions

Problem #1: Entropy

$$S = kT\left(\frac{\partial \ln \mathcal{Q}}{\partial T}\right)_{zV} - \bar{N}k\ln z + k\ln\mathcal{Q}$$

$$\ln\mathcal{Q} = \frac{1}{a}\sum_{\varepsilon}\ln\left(1+aze^{-\beta\varepsilon}\right)$$

$$S = k\sum_{\varepsilon}\left\{\frac{T}{a}\frac{\partial}{\partial T}\ln\left(1+aze^{-\beta\varepsilon}\right)\Big|_{z,V} - \langle n_\varepsilon\rangle\ln z + \frac{1}{a}\ln\left(1+aze^{-\beta\varepsilon}\right)\right\}$$

$$= k\sum_{\varepsilon}\left\{\frac{\varepsilon}{kT}\frac{1}{z^{-1}e^{\beta\varepsilon}+a} - \langle n_\varepsilon\rangle\ln z + \frac{1}{a}\ln\left(1+aze^{-\beta\varepsilon}\right)\right\}$$

$$\langle n_\varepsilon\rangle = \frac{1}{z^{-1}e^{\beta\varepsilon}+a} \quad\rightarrow\quad \beta\varepsilon - \ln z = \ln\frac{1-a\langle n_\varepsilon\rangle}{\langle n_\varepsilon\rangle}$$

$$1+aze^{-\beta\varepsilon} = \frac{1}{1-a\langle n_\varepsilon\rangle}$$

$$S = k\sum_{\varepsilon}\left\{\langle n_\varepsilon\rangle\ln\frac{1-a\langle n_\varepsilon\rangle}{\langle n_\varepsilon\rangle} + \left(-\frac{1}{a}\right)\ln\left(1-a\langle n_\varepsilon\rangle\right)\right\}$$

$$= k\sum_{\varepsilon}\left\{\left(\langle n_\varepsilon\rangle - \frac{1}{a}\right)\ln\left(1-a\langle n_\varepsilon\rangle\right) - \langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle\right\}$$

BE: $a=-1$

$$S = k\sum_{\varepsilon}\left\{\left(\langle n_\varepsilon\rangle+1\right)\ln\left(1+\langle n_\varepsilon\rangle\right) - \langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle\right\}$$

FD: $a=+1$

$$S = k\sum_{\varepsilon}\left\{-\left(1-\langle n_\varepsilon\rangle\right)\ln\left(1-\langle n_\varepsilon\rangle\right) - \langle n_\varepsilon\rangle\ln\langle n_\varepsilon\rangle\right\}$$

Problem #2:

(a) $$N = \frac{L^2}{(2\pi\hbar)^2}\int_0^\infty dp\, 2\pi p \frac{1}{z^{-1}e^{\beta p^2/2m}-1} + \frac{z}{1-z}$$

$$\frac{N}{L^2} = \frac{2\pi m kT}{h^2}\int_0^\infty dx \frac{1}{z^{-1}e^{x}-1} + \frac{1}{L^2}\frac{z}{1-z} = \frac{2\pi m kT}{h^2} g_1(z) + \frac{1}{L^2}\frac{z}{1-z} = n$$

if $z<1$ the fraction of bosons in the $\vec{p}=0$ state is zero in the thermodynamic limit

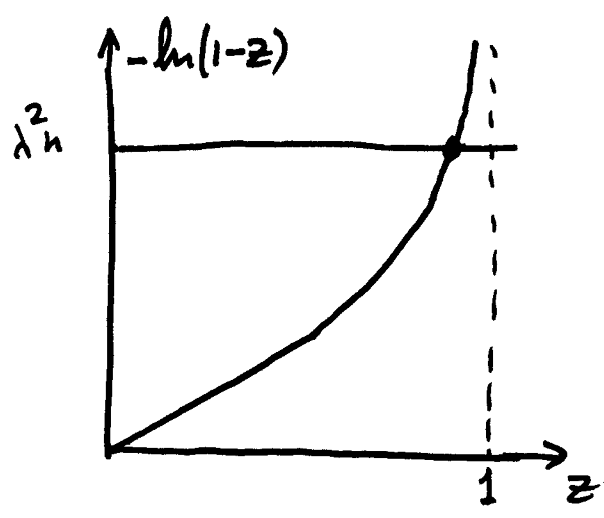
if $z\to 1$ the fraction of bosons in the $\vec{p}=0$ state is now finite.

(b) Evaluate $g_1(z)$: $z<1$

$$\int_0^\infty dx \frac{1}{z^{-1}e^{x}-1} = -\int_0^\infty dx \frac{e^{-x}}{e^{-x}-z^{-1}} = -\int_0^1 dy \frac{1}{y-z^{-1}} = -\ln(1-z) = g_1(z)$$

$$n = \frac{2\pi m kT}{h^2}\left(-\ln(1-z)\right) + \frac{1}{L^2}\frac{z}{1-z} \quad ; \quad \lambda^2 = \frac{h^2}{2\pi m kT} \text{ thermal length}$$

$$-\ln(1-z) = \lambda^2 n - \frac{\lambda^2}{L^2}\frac{z}{1-z}$$



In the thermodynamic limit ($L\to\infty$) and for $z<1$ there is only one solution for given $\lambda^2 n$. Hence, for $T\neq 0$, $\lambda^2 n$ is finite and $z<1$. No condensation for $T>0$.

(c) At $T=0$ all particles are in the $\vec{p}=0$ state. The ground state consists of all particles in the condensate! Hence, $T_c = 0$!

Problem #3: liquid He^3

density: $\rho = 0.081 \frac{g}{cm^3} = \frac{N}{V} m$

Fermi sphere: $N = 2 \sum_{\vec{k}\,occ}' = \frac{2V}{(2\pi)^3} 4\pi \int_0^{k_F} dk\, k^2 = \frac{V}{(2\pi)^3} \frac{8\pi}{3} k_F^3 = V \frac{k_F^3}{3\pi^2}$

$$k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} = \left(3\pi^2 n\right)^{1/3}$$

Fermi energy:
$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$
$$= \frac{(1.055\times 10^{-27}\, erg/s)^2}{2\times 3\times 1.67\times 10^{-24} g} \cdot \left(3\pi^2 \frac{\rho}{m}\right)^{2/3}$$
$$= \frac{(1.055\times 10^{-27}\, erg/s)^2}{2\times 3\times 1.67\times 10^{-24} g} \left(3\pi^2 \frac{0.081\, \cancel{g}/cm^3}{3\times 1.67\times 10^{-24} \cancel{g}}\right)^{2/3}$$
$$= 6.85\times 10^{-16}\, erg \simeq 0{,}43\, meV$$

Fermi temperature: $T_F = \frac{\varepsilon_F}{k_B} \simeq 5K$