STATISTICAL MECHANICS — PHY5524

Final Exam

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I have upheld the Academic Honor Code.

Best of Luck and Have a Wonderful Summer!

PROBLEM 1

Consider a system of N atoms in contact with a thermal reservoir at a temperature T. You may neglect the translational motion of the atoms and consider them as localized ("distinguishable") objects. Each hypothetical atom displays the following spectrum (see figure): A non-degenerate ground state with energy $E_0 = 0$ and charge $Q_0 = 0$. A first (single-ionized) state with an energy $E_1 = \Delta > 0$ and a charge (in units of the proton charge) of $Q_1 = 1$. Finally, a second (doubly-ionized) two-fold degenerate state with an energy $E_2 = 2\Delta$ and a charge (also in units of the proton charge) of $Q_2 = 2$.

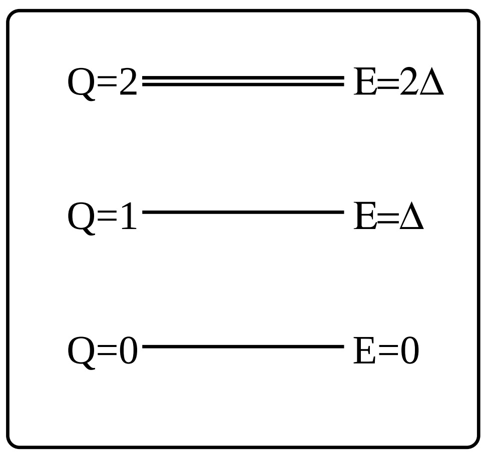
(a) Obtain the partition function and Helmholtz free energy of the system.

(b) Obtain the entropy of the system. Is your result consistent with the third law of thermodynamics?

(c) Obtain the energy of the system.

(d) Obtain the leading temperature-dependent term of the specific heat of the system in the low temperature ($k_B T \ll \Delta$) limit.

(e) Compute the temperature of the thermal reservoir required for the average charge of the system to be equal to $\langle Q \rangle = 1$.



PROBLEM 2

Consider a classical system of A identical non-interacting nucleons enclosed in a box of volume V and in contact with a thermal reservoir at a temperature T. The Hamiltonian of a single nucleon is given in terms of the coordinates and momenta of its three constituent quarks as follows:

$$H(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3; \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{2M}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{1}{2}K\left[(\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2 + (\mathbf{r}_3 - \mathbf{r}_1)^2\right] ,$$

where M is the mass of the constituent quarks and K is the "stiffness" of the harmonic (spring) coupling.

(a) Show that in terms of the coordinate of the center of mass ($\mathbf{R}$) and the two relative coordinates ($\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$), defined as

$$\mathbf{R} = \frac{1}{3}\left(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3\right) , \quad \boldsymbol{\rho} = \frac{1}{\sqrt{2}}\left(\mathbf{r}_1 - \mathbf{r}_2\right) , \quad \boldsymbol{\lambda} = \frac{1}{\sqrt{6}}\left(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3\right) ,$$

the single-nucleon Hamiltonian may be conveniently re-written in terms of a pure translation of the center of mass plus two uncoupled harmonic oscillators. That is,

$$H(\mathbf{P}, \mathbf{p}_\rho, \mathbf{p}_\lambda; \boldsymbol{\rho}, \boldsymbol{\lambda}) = \frac{P^2}{6M} + \frac{1}{2M}\left(p_\rho^2 + p_\lambda^2\right) + \frac{3K}{2}\left(\rho^2 + \lambda^2\right) .$$

(b) Obtain the partition function and Helmholtz free energy of the system. Is there a need for a Gibbs correction factor ($A!$)?

(c) Obtain the entropy of the system. Is your result consistent with the third law of thermodynamics?

(d) Obtain the energy of the system and the specific heat at constant volume.

(e) Obtain the equation of state of the system, *i.e.*, the pressure of the system as a function of A, T, and V.

PROBLEM 3

Consider an ideal gas of spin-1 bosons of mass M satisfying the conventional nonrelativistic dispersion relation

$$\epsilon(p) = \frac{p^2}{2M} \ .$$

The bosons are in contact with a reservoir at a temperature T, chemical potential μ, and are enclosed in a hypothetical 4-dimensional box of volume $V = L^4$.

(a) Obtain an expression for the average density of the system $\langle N \rangle / V \equiv N/V$ in terms of the temperature and chemical potential (or if you prefer, the fugacity $z = \exp(\beta\mu)$).

(b) Show in detail that when the temperature of the thermal reservoir falls below a critical temperature T_c, the system displays the phenomenon of Bose-Einstein condensation. Obtain an expression for the critical temperature T_c.

(c) Obtain an expression for the fraction of particles in the condensate N_0/N as a function of temperature for $T < T_c$.

(d) Obtain the equation of state of the system, *i.e.*, the pressure of the system as a function of N_0, N, T, and V for $T < T_c$.

You may find the following expressions useful:

1. The four-dimensional sum over momenta may be replaced (in the thermodynamic limit) by an integral of the form:

$$\sum_p \longrightarrow \frac{2\pi^2}{h^4} V \int_0^\infty p^3 dp \ .$$

2. The *Bose functions* $g_\nu(z)$ are defined as

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{z^{-1}e^x - 1} dx \ .$$