STATISTICAL MECHANICS — PHY5524

Final Exam
Instructor: Jorge Piekarewicz
(April 26, 2006)

DO NOT START WORKING ON THE EXAM UNTIL YOU HAVE READ CAREFULLY THE FOLLOWING INSTRUCTIONS. The final exam consists of three problems. You are required to solve only two out of the following three problems; the choice is yours not mine. Do **NOT** hand in three solutions! For the one problem that you have decided to drop, simply hand in a blank sheet of paper with your name and social security number on it along with the word **Drop**—and nothing else.

Academic Honor Code: *Students are expected to uphold the Academic Honor Code published in the Florida State University Bulletin and the Student Handbook. The first paragraph reads: The Academic Honor System of Florida State University is based on the premise that each student has the responsibility (1) to uphold the highest standards of academic integrity in the student's own work, (2) to refuse to tolerate violations of academic integrity in the University community, and (3) to foster a high sense of integrity and social responsibility on the part of the University community.*

Student's Name:
Student's Social Security Number:

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Student's Signature
I have upheld the Academic Honor Code.

Best of Luck and Have a Wonderful Summer!

PROBLEM 1

Consider a "rubber band" that is in contact with a thermal reservoir at a temperature T and is fastened at one end to a peg in the ceiling and supports from its other end an object of mass M. Assume a simple microscopic model of the rubber band as a linked chain of N segments joined end to end. Each of these segments has a fixed length a and can orient itself either parallel or anti-parallel to the vertical direction, as shown in the figure. You can neglect the mass of each of the segments and any interaction among them. Thus, the energy of the system equals the gravitational energy of the object $E = Mgh$, where the height h depends on the configuration (*i.e.*, microstate) of the segments.

(a) Obtain the partition function and the Helmholtz free energy of the system.

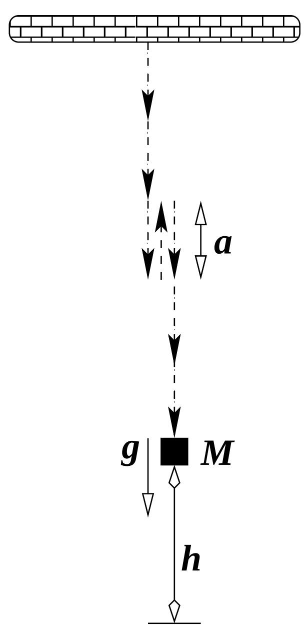
(b) Obtain the energy of the system and the average length of the rubber band.

(c) Obtain the entropy of the system. Is your result consistent with the third law of thermodynamics?

(d) Obtain the leading temperature-dependent term of the specific heat of the system in the low temperature ($k_B T \ll Mga$) limit.

You may find the binomial expansion useful:

$$(1+x)^N = \sum_{n=0}^{N} \binom{N}{n} x^n = \sum_{n=0}^{N} \left[\frac{N!}{n!(N-n)!}\right] x^n .$$



PROBLEM 2

Consider a classical ideal gas of N identical non-interacting particles of mass m enclosed in a **two-dimensional** box of area A that is in contact with a thermal reservoir at a temperature T. The particles obey the following fully-relativistic dispersion (*i.e.*, energy-vs-momentum) relation:

$$\epsilon(\mathbf{p}) = \sqrt{(pc)^2 + (mc^2)^2}\ ,$$

where $p = |\mathbf{p}|$, m is the rest mass of the particle, and c is the speed of light.

(a) Compute the one-particle partition function $\mathcal{Z}_1(T, A)$ by using (without proof) that the partition function of the whole system may be written as

$$\mathcal{Z}(N, T, A) \propto \Big[\mathcal{Z}_1(T, A)\Big]^N .$$

Is there a need for Gibbs correction factor? Explain.

(b) Compute the energy per particle of the system. Is your result consistent with the equipartition theorem? Explain briefly.

(c) Compute the entropy per particle of the system. Is your result consistent with the third law of thermodynamics? Explain briefly.

(d) Imagine that contact is now broken with the thermal reservoir and that the system is expanded adiabatically (*i.e.*, at constant entropy) to a final area A'. What will be the final temperature (T') of the system?

PROBLEM 3

Consider an ideal Fermi gas of N identical non-interacting spin 1/2 particles of mass m enclosed in a **two-dimensional** box of area A that is in contact with a thermal reservoir at a temperature T. The particles obey the following non-relativistic dispersion (*i.e.*, energy-vs-momentum) relation:

$$\epsilon(\mathbf{p}) = \frac{p^2}{2m} \,,$$

where $p = |\mathbf{p}|$.

(a) Compute the Fermi momentum, the corresponding Fermi energy, and Fermi temperature of the system. Recall that these are all zero-temperature ($T = 0$) properties of the system.

(b) Compute **exactly** the chemical potential μ as a function of the number density (N/A) and the temperature (T) of the system. For this part you should make no assumptions on the smallness of T.

(c) Verify that in the $T \equiv 0$ limit the above expression for the chemical potential equals the Fermi energy of part (a). Also from the above expression, compute the value of the chemical potential μ in the classical limit.

(d) Assume (without proof) that the specific heat of the system at low temperatures ($T \ll T_F$) is of the form

$$\frac{C_{\rm v}}{Nk_{\rm B}} = \gamma \frac{T}{T_{\rm F}} \,,$$

where γ is a dimensionless constant of order 1 and $T_{\rm F}$ is the Fermi temperature. Discuss in a clear short sentence why is the value of the specific heat much smaller than its classical value. That is, why is

$$\frac{C_{\rm v}}{Nk_{\rm B}} = \gamma \frac{T}{T_{\rm F}} \ll 1 \,.$$