Statistical Mechanics — PHY 5524

Final Exam April 25, 2008

1. (35 pts) Consider a gas of photons with dispersion $\omega = c|\vec{k}|$ where c is the speed of light. The photons are confined to a three-dimensional volume V and are in thermal equilibrium at temperature T. In what follows you may use, without proof, the fact that the density of states for this system, $g(\omega)$, defined so that $g(\omega)d\omega$ is the number of single-photon states with angular frequency between ω and $\omega + d\omega$, is given by

$$g(\omega) = \frac{V\omega^2}{\pi^2 c^3}.$$

(a) For this photon gas, obtain an expression for the grand potential $\Sigma = -k_B T \ln \mathcal{Z}$ where $\mathcal{Z}$ is the grand partition function (remember that the chemical potential for photons is zero). This expression will involve an integral over frequency which you should **not** try to evaluate.

(b) Show that the dependence of Σ on volume V and temperature T is of the form $\Sigma \propto VT^\alpha$ and determine the exponent α.

(c) Show that the dependence of the entropy of this gas S on volume V and temperature T is of the form $S \propto VT^\beta$, and determine the exponent β.

(d) Now assume this gas undergoes an adiabatic (i.e. $\Delta S = 0$) expansion with initial temperature $T_i = 3000$ K and final temperature $T_f = 3$ K. By what factor does the volume of the gas increase? (Your result gives a good estimate for the factor by which the volume of the universe has increased since the photons which form the cosmic microwave background, which today has a temperature of $T \simeq 3$ K, decoupled from matter roughly 380,000 yrs after the big bang, when the temperature was $T \simeq 3000$ K.)

2. (35 pts) Consider a one-dimensional gas of N noninteracting spin-1/2 fermions with dispersion $\mathcal{E}(k) = \hbar^2 k^2/2m$ confined to a system of length L. The number of particles per unit length is then $n = N/L$.

(a) Obtain expressions for the Fermi wave vector k_F and the Fermi energy $\mathcal{E}_F$ as a function of n for this gas.

(b) Determine the one-particle density of states for this gas.

(c) Show that the total energy of this gas at $T = 0$ is given by $E = qN\mathcal{E}_F$ and determine the dimensionless number q.

(d) Use the fact that at $T = 0$ the pressure P (which in one dimension has units of force) is given by $P = -\left(\partial E/\partial L\right)_N$ to show that $P \propto n^\alpha$ at $T = 0$ and determine the exponent α.

3. (30 pts) Consider a system of N noninteracting bosons in a volume V. Assume the one-particle density of states for these bosons is given by

$$a(\mathcal{E}) = \begin{cases} gV\mathcal{E}^\nu & \mathcal{E} \geq 0, \\ 0 & \mathcal{E} < 0, \end{cases}$$

where g and ν are constants with $\nu > 0$. Show that this gas exhibits Bose condensation with a critical temperature $T_c \propto n^\delta$ where $n = N/V$ is the number density of bosons and determine the exponent δ.