

Classical Mechanics Solutions

Solution 1

Conservation of energy given by the sum of potential energy due to gravity and kinetic energy can be used to determine escape velocity. In the case of Earth along the potential is given by:

$$\phi(r) = -G \frac{M_E m}{r}$$

where m is the mass of the book. The book will escape if initial kinetic energy is high enough to overcome the potential at $r = R_E$. Thus

$$\frac{mv_E^2}{2} = G \frac{M_E m}{R_E} \text{ thus } v_E = \sqrt{\frac{2GM_E}{R_E}} = 11 \text{ km/s}$$

In the Earth-Moon case the potential is

$$\phi(r) = -G \frac{M_E m}{r} - G \frac{M_M m}{|R_{EM} - r|}$$

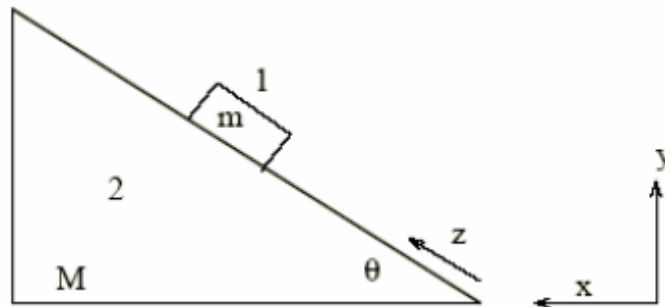
where $MM = ME$. The potential is a symmetric double-well and in order to leave the surface of the earth the kinetic energy must be high enough to overcome a saddle point right in the middle between earth and moon. Thus the condition for escape velocity is

$$\frac{mv_E^2}{2} - GM_E m \left(\frac{1}{R_E} + \frac{1}{R_{EM} - R_E} \right) = -\frac{4GM_E m}{R_{ME}}$$

This equation solved for escape velocity gives $v_E = 7.7 \text{ km/s}$.

Solution 2

Introduce the generalized coordinates as in the figure below.



The Lagrangian for this system will be given by

$$L = T_{\text{wedge}} + T_m - V_{\text{wedge}} - V_m \tag{1}$$

and

$$T_{wedge} = \frac{1}{2} M \dot{x}_2^2 \quad (2)$$

$$V_{wedge} = M g y_2 = \text{constant, since the wedge does not move in the } y \text{ direction.} \quad (3)$$

$$T_m = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2) \quad (4)$$

$$V_m = m g y_1 \quad (5)$$

But x_1 and y_1 can be written as follows:

$$x_1 = x_2 + z_1 \cos(\theta) \quad (6)$$

$$y_1 = z_1 \sin(\theta) \quad (7)$$

Substitute Eqs.6 and 7 in Eqs.4 and 5, then replace T's and V's in Eq.1, and rename $x_2 = x$, $z_1 = z$ for simplicity. The result is given by:

$$L = \frac{1}{2} m [\dot{x}^2 + \dot{z}^2 + 2\dot{x}\dot{z} \cos(\theta)] + \frac{1}{2} M \dot{x}^2 - m g \sin(\theta) \quad (8)$$

The equations of motion are obtained by:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0, \text{ where } q = x, z. \quad (9)$$

We get:

$$m\ddot{x} + m\ddot{z} \cos(\theta) + M\ddot{x} = 0 \quad (10)$$

$$m\ddot{z} + m\ddot{x} \cos(\theta) + m g \sin(\theta) = 0 \quad (11)$$

From Eq.11 solve for \ddot{z} , and then substitute in Eq.10

$$\ddot{x} = \frac{m g \sin(\theta) \cos(\theta)}{m + M - \cos^2(\theta)}. \quad (12)$$

Substituting the numerical values for different parameters, we get:

$$\ddot{x} = 1.9624 \frac{m}{s^2} \quad (13)$$

$$\ddot{z} = -6.82 \frac{m}{s^2} \quad (14)$$

To find the acceleration of the mass m , derivate Eqs.6 and 7 twice and use the numerical values from Eq13 and 14

$$\ddot{x}_1 = -3.85 \frac{m}{s^2} \quad (15)$$

$$\ddot{y}_1 = -3.58 \frac{m}{s^2} \quad (16)$$

Solution 3

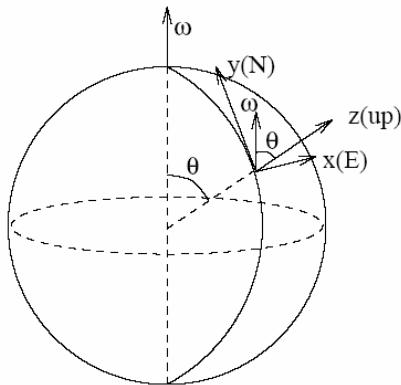
We use energy conservation for this problem, so that the initial potential energy $E = mg(L/2)$ becomes kinetic energy of rotation $E = I\omega^2 / 2$. The rotation of the pencil takes place around its tip, thus the moment of inertia is:

$$I = \frac{M}{L} \int_0^L x^2 dx = M \frac{L^3}{3}.$$

The velocity of the eraser is related to the angular velocity as $v = \omega L$. From here we get $v = \sqrt{3gL}$ which can be compared with the free fall speed of $v_{free} = \sqrt{2gL}$.

Solution 4

Referring to the diagram below, we see that in this coordinate frame the vector $\boldsymbol{\omega}$ has a component $\omega \cos(\theta)$ in the up direction and a component $\omega \sin(\theta)$ in the North direction. Suppose that the velocity vector relative to the moving coordinate frame is $\mathbf{v}_r = (0, 0, \dot{z})$. Then assuming $\dot{z} > 0$ the vector $\boldsymbol{\omega} \times \mathbf{v}_r$ points East (along $\hat{\mathbf{x}}$) and has magnitude $\omega \dot{z} \sin(\theta)$, so the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}_r$ points West with magnitude $2m\omega \dot{z} \sin(\theta)$.



If we combine this with a gravitational force pointing down (along $-\hat{\mathbf{z}}$), the equations of motion become

$$m\ddot{x} = -2m\omega \dot{z} \sin(\theta)$$

$$\begin{aligned} m\ddot{y} &= 0 \\ m\ddot{z} &= -mg. \end{aligned}$$

b) The particle is dropped at rest from a height h above the ground; it arrives at the ground with velocity $v_0 = \sqrt{2gh}$. Find the magnitude and direction of the Coriolis deflection.

Because the Coriolis force is so weak compared to the gravitational force [ω is $2\pi/(24 \cdot 3600) = 7.27 \cdot 10^{-5}$], we can safely assume that during the entire flight we can continue to neglect the Coriolis forces in the \hat{y} and \hat{z} directions. First solve the z equation of motion to find

$$\dot{z}(t) = -gt$$

and then substitute \dot{z} into the x equation to find

$$\ddot{x} = -2\omega \sin(\theta)(-gt) = 2\omega \sin(\theta)gt.$$

Integrating this over time twice and using the initial conditions $\dot{x}(0) = x(0) = 0$, we find

$$\begin{aligned} \dot{x} &= \dot{x}(0) + 2\omega \sin(\theta)g \frac{t^2}{2} \\ x(t) &= x(0) + \omega \sin(\theta)g \frac{t^3}{3}. \end{aligned}$$

To find the Coriolis deflection we need to find the time at which the particle hits the ground, which is (to a good approximation) $t_f = \sqrt{2h/g}$, just an object falling in constant gravity. The deflection is therefore towards the East (along \hat{x}) with magnitude

$$x(t_f) = \frac{1}{3}\omega \sin(\theta)g \left(\frac{2h}{g}\right)^{\frac{3}{2}} = \frac{1}{3}\omega \sin(\theta)\sqrt{\frac{8h^3}{g}}$$

c) The particle is now thrown vertically upward with an initial speed v_0 , so that it reaches the maximum height h [the same h as in part (b)], and then it falls back to the ground. Find the magnitude and direction of the Coriolis deflection.

This goes exactly as before, except now the vertical velocity is given by

$$\dot{z} = v_0 - gt,$$

where $v_0 = \sqrt{2gh}$ is the velocity required to get the particle to height h if it is thrown vertically upward. Substituting this into the x equation as before we get

$$\ddot{x} = -2\omega \sin(\theta)[v_0 - gt].$$

Integrating this over time twice and again using the initial conditions $\dot{x}(0) = x(0) = 0$, we find

$$\dot{x} = \dot{x}(0) - 2\omega \sin(\theta) \left[v_0 t - g \frac{t^2}{2} \right]$$

$$\begin{aligned}
 x(t) &= x(0) - 2\omega \sin(\theta) \left[v_0 \frac{t^2}{2} - g \frac{t^3}{6} \right] \\
 x(t) &= -\omega \sin(\theta) \left[v_0 t^2 - g \frac{t^3}{3} \right]
 \end{aligned}$$

Now we have to find the time of flight, which is simply twice the value we had before, $t_f = 2\sqrt{2h/g}$, so that the final x displacement is

$$\begin{aligned}
 x(t_f) &= -\omega \sin(\theta) \left[\sqrt{2gh} t_f^2 - g \frac{t_f^3}{3} \right] \\
 &= -\omega \sin(\theta) \left[\sqrt{2gh} \frac{8h}{g} - g \frac{16h}{3g} \sqrt{\frac{2h}{g}} \right] \\
 &= -\omega \sin(\theta) \left[4 - \frac{8}{3} \right] \sqrt{\frac{8h^3}{g}} \\
 &= -\frac{4}{3} \omega \sin(\theta) \sqrt{\frac{8h^3}{g}},
 \end{aligned}$$

where the negative sign means that the displacement is to the West.

d) Compare your results for parts (b) and (c).

The direction of the deflection is opposite and it is four times as large.