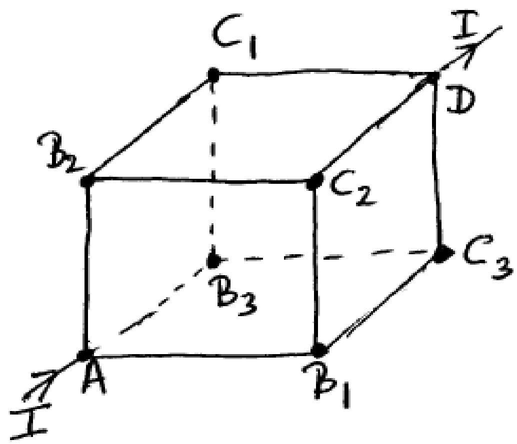
Electrodynamics Solutions

Solution 1



-The AD-axis of the cube has threefold symmetry, i.e. the cube is invariant under rotations by $\pm 120^0$ about that axis.
-Hence, the corners B_1, B_2 and B_3 are equivalent and have the same potential.
-Also, the corners C_1, C_2 and C_{3_3} are equivalent and have the same potential.

If a current I enters A, then:

- A current $\frac{I}{3}$ circulates through each of the resistors AB_i.
- A current $\frac{I}{6}$ circulates through each of the six resistors B_iC_j.
- A current $\frac{I}{3}$ passes through each of the three resistors C_jD.
- The potential drop between A and B_i is $\frac{rI}{3}$.
- The potential drop between B_i and C_j is $\frac{rI}{6}$.
- The potential drop between C_j and D is $\frac{rI}{6}$.

The potential difference between A and D is:

$$rI\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{3}\right)=\frac{5}{6}rI$$

Hence $R=\frac{5}{6}r$

Solution 2

a) Calculate the capacitance of the capacitor.
To find the capacitance we would like to find the voltage between the plates for a given charge +Q and −Q on the two plates. We can use Gauss's law (both with and without a dielectric) to find the electric field due to the charge on the plates, using the approximation that there are no end effects. Gauss's law reads, in general

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enclosed}}{\varepsilon}$$

where $\varepsilon = \kappa\varepsilon_o$ is the general form of the electric permittivity in the presence of a dielectric (note that in free space $\kappa = 1$).
If we use a pill-box shaped Gaussian surface with end area A′ and with one end below the bottom plate, which is charged to −Q, and the other end inside the dielectric, then Gauss's law gives us

$$\oint \vec{E}_d \cdot d\vec{S} = E_d A' = \frac{Q_{enc}}{\varepsilon} = \frac{\sigma A'}{\varepsilon} = \frac{Q}{\varepsilon A} A',$$

where Ed is the field inside the dielectric, since the field is zero outside the electric on the bottom end of the pill-box and always perpendicular to the sides of the pill-box. This gives

$$E_d = \frac{Q}{\varepsilon A} = \frac{Q}{\kappa\varepsilon_o A}.$$

Exactly the same procedure, but with the two ends of the pillbox above and below the top plate in empty space, results in the field between the dielectric and the top plate of

$$E = \frac{Q}{\varepsilon_o A} = \frac{Q}{\varepsilon_0 A}.$$

We can now find the relation between the voltage difference V between the bottom and top plates and Q by integrating the constant electric field (which points straight up) along a vertical line from the bottom plate to the top plate

$$V = \int_0^{L/2} \vec{E}_d \cdot d\vec{l} + \int_{L/2}^{L} \vec{E} \cdot d\vec{l} = \frac{Q}{\kappa\varepsilon_0 A}\frac{L}{2} + \frac{Q}{\varepsilon_0 A}\frac{L}{2} =$$

so that we have

$$C = \frac{Q}{V} = \frac{2\varepsilon_o A}{L}\left(\frac{1}{\kappa} + 1\right)^{-1} = \frac{2\varepsilon_o A}{L}\left(\frac{\kappa}{1+\kappa}\right).$$

Note that this is just the capacitance $\frac{2\varepsilon_o A}{L}$ of a free-space parallel-plate capacitor with area A and plate gap L/2 in series with that of the same capacitor with dielectric inserted, which is larger, $2\kappa\varepsilon_0 AL$. The total capacitance of two capacitors in series is like that of resistors in parallel, i.e. it is given by

$$C = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}.$$

b) Calculate the charge on the capacitor.

For any capacitor Q = CV, so

$$Q = \frac{2\varepsilon_0 AV}{L}\left(\frac{\kappa}{1+\kappa}\right).$$

c) Calculate the value of the electric displacement D in the capacitor.
The source of the electric displacement D is free charges, so we are to ignore any surface charges on the dielectric. This means that the electric displacement everywhere in the capacitor is just the electric field in the free space region,

$$D = E = \frac{Q}{\varepsilon_o A} = \frac{2V}{L}\left(\frac{\kappa}{1+\kappa}\right)$$

and it points from the lower plate to the upper plate.

d) Calculate the value of the electric field inside the dielectric layer, and in the air above it. With our approach we have already done this and we have

$$E_d = \frac{Q}{\kappa\varepsilon_o A} = \frac{2V}{L(1+\kappa)}, \qquad E = \frac{Q}{\varepsilon_0 A} = D.$$

e) Calculate the electrostatic energy stored in the system. How would it change if the dielectric is removed? The electrostatic energy stored in a capacitor is just

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

and so we see that

$$U = \frac{1}{2}\frac{2\varepsilon_o A}{L}\left(\frac{\kappa}{1+\kappa}\right)V^2 = \frac{\varepsilon_o AV^2}{L}\left(\frac{\kappa}{1+\kappa}\right).$$

Obviously if κ is reduced to unity by removing the dielectric but the applied voltage remains the same, the capacitance is reduced and so the amount of stored energy is reduced.

Solution 3

The flux Φ is given by

$$\Phi = BA = \begin{cases} ktA & t < t_0 \\ kt_0 A & t > t_0 \end{cases}$$

Therefore we have

$$\varepsilon = -\frac{d\Phi}{dt} = \begin{cases} kA & t < t_0 \\ 0- & t > t_0 \end{cases}$$

where the minus sign reminds us that the current flows to oppose the increasing B, or in the clockwise direction. The total voltage Kirchoff equation is

$$V_R + V_L = |\varepsilon|$$

$$IR + L\frac{dI}{dt} = |\varepsilon|$$

$$\text{Or } \frac{dI}{dt} = |\varepsilon| - \frac{IR}{L}$$

The general solution of this differential equation may easily be obtained by the substitution $u = |\varepsilon| - \frac{IR}{L}$, which yields $u = u_0 \exp(-\frac{Rt}{L})$ where $u_0 = |\varepsilon|$, and hence $I = \frac{|\varepsilon|}{R}\left(1 - u_0 \exp\left(-\frac{Rt}{L}\right)\right)$.

Or, following the conventions of the problems class:

We can solve this by adding a particular solution $I_P = \frac{|\varepsilon|}{R}$ to the solution $I_h = k\exp\left(-\frac{Rt}{L}\right)$ of the homogenous equation $L\frac{dI}{dt} = -RI$, and then matching to initial conditions which are I(t = 0) = 0, we have:

$$I = \frac{|\varepsilon|}{R} + k\exp\left(-\frac{Rt}{L}\right)$$

$$I(0) = \frac{|\varepsilon|}{R} + k \Rightarrow k = -\frac{|\varepsilon|}{R}$$

$$I(t) = \frac{kA}{R}\left(1 - \exp\left(-\frac{Rt}{L}\right)\right), \quad 0 < \mathrm{t} < t_0$$

For t > t_0 there is only an exponential decay of the current $I(t_0) = \frac{KA}{R}\left(1 - \exp\left(-\frac{Rt_0}{L}\right)\right)$.

This time the differential equation is:

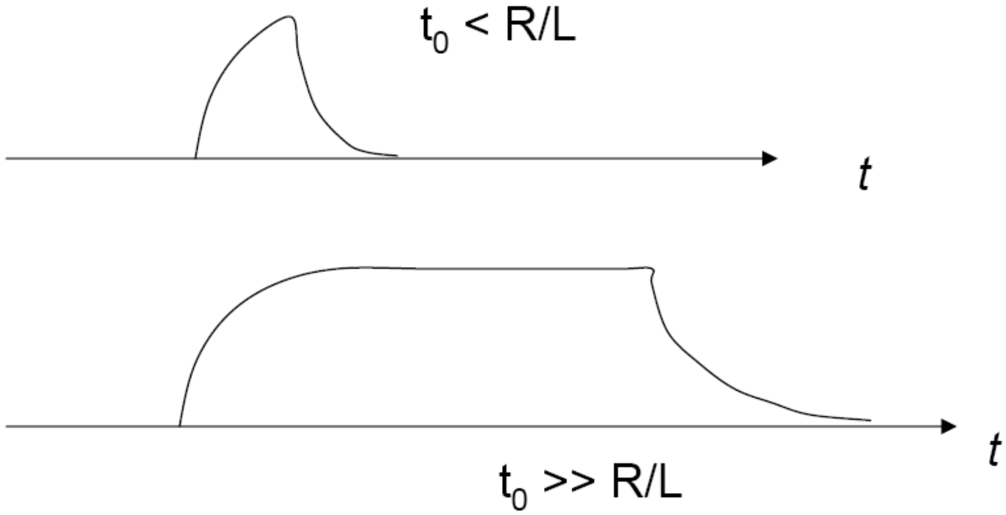
$$\frac{dI}{dt} = -\frac{IR}{L}$$

With the immediate solution $I = I_0 \exp\left(-\frac{(t - t_0)R}{L}\right)$ since now the limits of the integration are from I(t_0) to I(t) and t_0 to t.

Hence we have.

$$I(t) = \frac{KA}{R}\left(1 - \exp\left(-\frac{Rt_0}{L}\right)\right)\exp\left(-\frac{R(t - t_0)}{L}\right)$$

$$I(t) = \frac{KA}{R}\left(\exp\left(\frac{Rt_0}{L}\right) - 1\right)\exp\left(-\frac{Rt}{L}\right)$$



Solution 4

Solution[1]

- Using divergence theorem we find

$$4\pi\rho = \nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = A(1 - br)\,\frac{\exp(-br)}{r^2},$$

  thus, charge density as a function of radius is

$$\rho(r) = A(1 - br)\,\frac{\exp(-br)}{4\pi r^2}.$$

  The charge density behaves as A/r^2 near origin, becomes zero at $r = 1/b$, and exponentially decays to zero at large distance as $-Ab\,\exp(-br)/r$.

- The total charge is given by an integral

$$Q = 4\pi \int r^2 \rho dr,$$

  its calculation is unnecessary since from the Gauss law for spherically symmetric distribution we know

$$Q = \lim_{r\to\infty} r^2 E_r = 0.$$

Solution 5

We solve this problem by the principle of superposition. The magnetic field is identical to that of a solid conductor of radius R with the same current density as the cylindrical conductor with the hole in it, superimposed on another solid conductor with radius a and the same magnitude of the current density but with the current in the opposite direction at the position of the hole. First let's find the current in each. The current density is $J = I/[\pi(R^2 - a^2)]$, so that the magnitude of the current in the first solid conductor should be

$$I_R = J \cdot \pi R^2 = I \frac{R^2}{R^2 - a^2},$$

and that in the second conductor should be

$$I_a = J \cdot \pi a^2 = I \frac{a^2}{R^2 - a^2}.$$

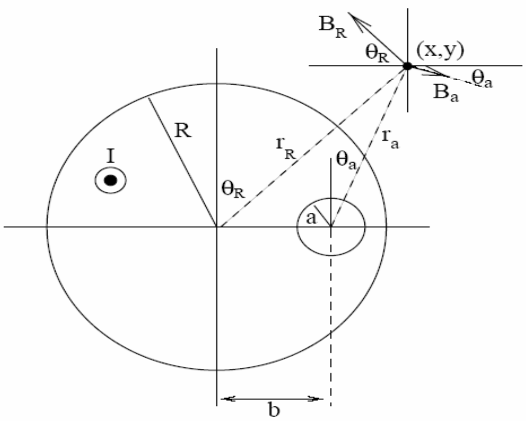
Now we can use Ampère's law to find the magnetic field at a general point $P = (x, y)$ outside the two conductors, because each is now a symmetric conductor and we can do the integral of the magnetic field around a circle centered at the center of each wire. Formally, we have

$$\begin{aligned} \oint \mathbf{B}_R \cdot \mathbf{dl} &= \mu_0 I_R \\ \oint \mathbf{B}_a \cdot \mathbf{dl} &= \mu_0 I_a \end{aligned}$$

where the integrals are over circles centered at the center of the solid conductor of radius R and that of radius a, respectively. This means that

$$\begin{aligned} B_R &= \frac{\mu_0}{2\pi} \frac{I_R}{r_R} \\ B_a &= \frac{\mu_0}{2\pi} \frac{I_a}{r_a}, \end{aligned}$$

where r_R and r_a are the distances to the centers of the two conductors from the field point $P = (x, y)$. The direction of each $\mathbf{B}$ is different; the large conductor has a current out of the page so its field points along $\hat{\boldsymbol{\theta}}_R$, and that from the small conductor points along $-\hat{\boldsymbol{\theta}}_a$, as shown below.



From the diagram we can see that

$$\begin{aligned} B_{Rx} &= -B_R \cos(\theta_R) = -\frac{\mu_0}{2\pi} \frac{I_R}{\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} = -\frac{\mu_0}{2\pi} \frac{I_R\, y}{x^2+y^2} \\ B_{Ry} &= B_R \sin(\theta_R) = \frac{\mu_0}{2\pi} \frac{I_R}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}} = \frac{\mu_0}{2\pi} \frac{I_R\, x}{x^2+y^2} \\ B_{ax} &= B_a \cos(\theta_a) = \frac{\mu_0}{2\pi} \frac{I_a}{\sqrt{(x-b)^2+y^2}} \frac{y}{\sqrt{(x-b)^2+y^2}} = \frac{\mu_0}{2\pi} \frac{I_a\, y}{(x-b)^2+y^2} \\ B_{ay} &= -B_a \sin(\theta_a) = -\frac{\mu_0}{2\pi} \frac{I_a}{\sqrt{(x-b)^2+y^2}} \frac{(x-b)}{\sqrt{(x-b)^2+y^2}} = -\frac{\mu_0}{2\pi} \frac{I_R\,(x-b)}{(x-b)^2+y^2}, \end{aligned}$$

so that

$$\begin{aligned} B_x &= B_{Rx} + B_{ax} = -\frac{\mu_0}{2\pi} I y \left(\frac{R^2}{R^2-a^2} \frac{1}{x^2+y^2} - \frac{a^2}{R^2-a^2} \frac{1}{(x-b)^2+y^2} \right) \\ B_y &= B_{Ry} + B_{ay} = -\frac{\mu_0}{2\pi} I \left(\frac{R^2}{R^2-a^2} \frac{x}{x^2+y^2} - \frac{a^2}{R^2-a^2} \frac{x-b}{(x-b)^2+y^2} \right). \end{aligned}$$

Solution 6

Kinetic Energy of rotation: $KE = \frac{1}{2} I_Z \omega^2$

$I_Z = \frac{1}{2} Mr^2$, $M = 2\pi Ar\rho$, $I_Z = \pi Ar^3 \rho$

r= radius, A= cross section, ρ= mass density

The magnetic flux through a ring is: $\Phi = H_o r^2 \ \pi \cos(\theta)$

θ= the angle between the field and the plane of the ring

The rotation of the ring: θ=ωt, (assuming ω>>1/τ)

Lenz's law: $V = -\frac{1}{c}\frac{d\phi}{dt} = \frac{r^2 \pi H_o \omega}{c} \sin(\omega t)$

Again, assuming that ω varies only slowly with time.

Electric dissipation: $IV = \frac{V^2}{R} = -\frac{d}{dt}(\frac{1}{2} I_z \omega^2) = -I_z \omega \dot{\omega}$

Here, $R = \frac{2r\pi}{A\sigma}$ and, hence: $I_z \omega \dot{\omega} = -\frac{1}{R}\frac{(r^2 \pi H_o \omega)^2}{c^2} \sin^2(\omega t)$

Averaging over short times $\langle \dot{\omega} \rangle = -\frac{\omega}{\tau}$, and $\langle \sin^2(\omega t) \rangle = \frac{1}{2}$

$$\frac{1}{\tau} = \frac{1}{RI_z}\frac{(r^2 \pi H_o)^2}{2c^2} = \frac{(r^2 \pi H_o)^2 A\sigma}{2r\pi r^3 \pi A\rho 2c^2} = \frac{\sigma H_o^2}{4\rho c^2}$$

Then: $\tau = \frac{4\rho c^2}{\sigma H_o} = \frac{4 \cdot (8.9) \cdot (9 \cdot 10^{20}) gcm^{-3} cm^{2/s^2}}{(5 \cdot 10^{17}) s^{-1} (4 \cdot 10^4) G^2} = 1.6s$

For $G^2 = erg / cm^3 = \frac{gcm^2}{s^2 cm^3}$