

## Quantum Mechanics Problem 1

Consider the three spin-1 matrices

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

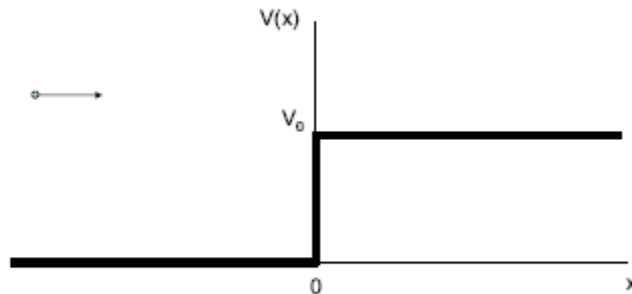
- (a) Calculate the commutator of  $S_x$  and  $S_y$ .
- (b) What are the possible values we can get if we measure the spin along the x-axis?
- (c) Suppose we obtain the largest possible value when we measure the spin along the x-axis. If we now measure the spin along the z-axis, what are the probabilities for the various outcomes?

## Quantum Mechanics Problem 2

Consider a one-dimensional step potential of the form:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

where  $V_0 > 0$ . A quantum particle with mass  $m$  and energy  $E > V_0$  is incident on this step “from the left” as shown in the figure.



- (a) Write down the appropriate solutions of the time-independent Schrödinger equation for this particle in the  $x < 0$  region and the  $x > 0$  region.

- (b) Apply the appropriate boundary conditions at the point  $x = 0$  to match these solutions.
- (c) Derive expressions for the probabilities that the particle is reflected (R) and transmitted (T) by the step.

**Hint:** Recall that the probability density current is given by:

$$j(x) = \frac{\hbar}{2im} \left( \Psi^* \frac{\partial \Psi}{\partial x} - \Psi \frac{\partial \Psi^*}{\partial x} \right).$$

### Quantum Mechanics Problem 3

The time-independent wave function for a particle of mass  $m$  which moves in a one-dimensional potential  $V(x)$  has the form  $\psi(x) = A \exp[-a^2 x^2]$ , where  $a = \sqrt{m\omega/2\hbar}$  and  $A$  is normalization constant.

- a. Using the time-independent Schrodinger equation find  $V(x)$  and the energy eigenvalue for  $\psi(x)$ .
- b. Identify the system. Which one of its quantum states is described by  $\psi(x)$ ?

### Quantum Mechanics Problem 4

Consider two solutions to the one-dimensional time-independent Schrödinger equation with same energy  $E$ :  $\psi_1(x)$  and  $\psi_2(x)$ .

- (a) Prove that regardless of the potential function  $V(x)$ ,

$$\psi_1 \frac{d}{dx} \psi_2 - \psi_2 \frac{d}{dx} \psi_1 = C$$

where  $C$  is a constant.

- (b) Show that if  $\psi_1(x)$  and  $\psi_2(x)$  are bound state solutions, then  $C = 0$ .

From this, show that  $\psi_1 = \gamma \psi_2$  for some constant  $\gamma$ , thus proving that there are no degenerate bound state solutions to the one-dimensional time-independent Schrödinger equation.

- (c) Why does this theorem fail for continuum eigenstates (i.e. unbound states)? Give a specific example of degenerate continuum eigenstates in one dimension.

## Quantum Mechanics Problem 5

Consider a quantum mechanical system with two states,  $|\alpha\rangle$  and  $|\beta\rangle$ . In this orthonormal basis of states the Hamiltonian is given by the matrix

$$\begin{pmatrix} W & V \\ V & -W \end{pmatrix}.$$

- (a) Obtain the exact energy eigenvalues.  
 (b) Consider the Hamiltonian as  $H = H_w + H_v$ , where

$$H_w = \begin{pmatrix} W & 0 \\ 0 & -W \end{pmatrix}, \quad H_v = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix}$$

and assuming that  $|V| \ll |W|$  obtain the energy eigenvalues to second order in perturbation theory in  $V$ .

- (c) Compare your results in (a) and (b) and verify they agree to second order in  $V$ .

## Quantum Mechanics Problem 6

Consider the hyperfine splitting of the ground state of Hydrogen in a uniform magnetic field  $B$ . Assume the Zeeman energy shift is of the same order of magnitude as the hyperfine shift. The perturbation Hamiltonian is

$$V = A \vec{s}_e \cdot \vec{s}_p - \vec{\mu}_e \cdot \vec{B}$$

where  $\vec{\mu}_e = (e/m_e c) \vec{s}_e$ . The sub-indices  $e$  and  $p$  refer to the electron and proton, respectively. Using degenerate perturbation theory, find the energy shifts of all four of the  $n = 1, l = 0$  levels. You may ignore the  $-\vec{\mu}_p \cdot \vec{B}$  term.