Thermodynamic and Statistical Mechanics Solutions

Solution 1

(a) If n is the number of segments which point up and L is the length of the rubber band, then:

$$\frac{L}{a} = N - 2n \Rightarrow n = \frac{N}{2} - \frac{L}{2a}$$

The number of microstates corresponding to the length L is then N choose n, i.e.

$$\#\text{ of } microstates = \binom{N}{n} = \frac{N!}{(N-n)!n!} = \frac{N!}{\left(\frac{N}{2}+\frac{L}{2a}\right)!\left(\frac{N}{2}-\frac{L}{2a}\right)!}$$

From this, we find the entropy is:

$$S = k_B \ln \frac{N!}{\left(\frac{N}{2}+\frac{L}{2a}\right)!\left(\frac{N}{2}-\frac{L}{2a}\right)!}$$

(b) The energy of the system is simply $E(L) = -mgL$. Thus, the free energy at temperature T is:

$$F = -mgL - k_B T \ln \frac{N!}{\left(\frac{N}{2}+\frac{L}{2a}\right)!\left(\frac{N}{2}-\frac{L}{2a}\right)!}$$

(c) For $N >> 1$, Stirling's approximation tells us that:

$$\frac{S}{k_B} \cong N \ln N - \left(\frac{N}{2}+\frac{L}{2a}\right)\ln\left(\frac{N}{2}+\frac{L}{2a}\right) - \left(\frac{N}{2}-\frac{L}{2a}\right)\ln\left(\frac{N}{2}-\frac{L}{2a}\right)$$

Thus

$$\frac{\partial S}{\partial L} = \frac{k_B}{2a}\ln\frac{Na-L}{Na+L}$$

So, minimizing the Free energy we have

$$\frac{\partial F}{\partial L} = -mg - k_B T \frac{1}{2a}\ln\frac{Na-L}{Na+L} = 0$$

and, solving for L yields

$$L = Na \tanh \frac{mga}{k_B T}$$

(d) As T increases the rubber band gets shorter. This is seen directly from the solution of Part (c), but also from the fact that the higher temperatures favor higher entropy configurations and hence smaller rubber band lengths.

Solution 2

(a) The abrupt accumulation of bosons in the ground state at temperatures below T_C is called Bose-Einstein condensation. This is a property of bosons which have no restriction to the number of particles per energy state (as opposed to fermions which must obey the Exclusion Principle.)

(b) This can be solved in a few different ways. Here is the solution that the students have access to (which is correct):

For Bosons:

$$N = \sum_p \frac{1}{e^{\beta(E-\mu)}-1} = \frac{A}{(2\pi)^2}\int d^2p \frac{1}{e^{\beta(E-\mu)}-1}$$

Bose-Einstein condensation occurs if N is finite for $\mu \to 0$. At small energies (and therefore small momenta) the integral becomes:

$$\int \frac{d^2p}{e^{\beta E}-1} = \int \frac{pdp}{1+\beta E-1} = \int \frac{pdp}{\beta c p^{3/2}} \sim \int \frac{dp}{p^{1/2}}$$

Since this integral is finite (over all momenta), we have Bose condensation.

To calculate T_C, we have to do the integral:

$$N = \frac{A}{(2\pi)^2}\int d^2p \frac{1}{e^{\beta c p^{3/2}}-1}$$

Change of variable: let $x = \beta c p^{3/2}$

Therefore: $p = (kT_C / c)^{2/3} x^{2/3}$

Our integral is now:

$$N = \frac{A}{(2\pi)^2}\left(\frac{kT_C}{c}\right)^{4/3}\left(\frac{2}{3}\right)\int dx \frac{x^{1/3}}{e^x-1}$$

The integral just gives a number, so we have $N/A \sim (T_C)^{4/3}$ or $T_C \sim n^{3/4}$.

The exponent α=3/4.

(c) The differential of the grand potential is:

$$D\theta=-SdT-PdA+nd\mu$$

From this, we can calculate the entropy and pressure:

$$S = -\left(\frac{d\theta}{dT}\right)_{A,\mu}$$

$$P = -\left(\frac{d\theta}{dA}\right)_{T,\mu}$$

We can calculate the grand potential from:

$$\theta = -kT\ln Z$$

And the grand partition function:

$$Z = \sum e^{-\beta(E-\mu N)}$$

The grand partition function is a sum over particles and momenta:

$$Z = \prod_p \sum_n e^{-\beta(E-\mu)n} = \prod_p \frac{1}{1-e^{-\beta(E-\mu)}}$$

Calculating the grand potential:

$$\theta = kT\sum_p \ln(1-e^{-\beta(E-\mu)}) = \frac{kTA}{(2\pi)^2}\int d^2p \ln(1-e^{-\beta(E-\mu)})$$

Below T_C , μ=0. We can evaluate the integral through a change of variables:

$$p = \left(\frac{kT}{c}\right)^{2/3} x^{2/3}$$

Therefore:

$$\theta = kT(T)^{4/3}A(\text{int}) \sim T^{7/3}A$$

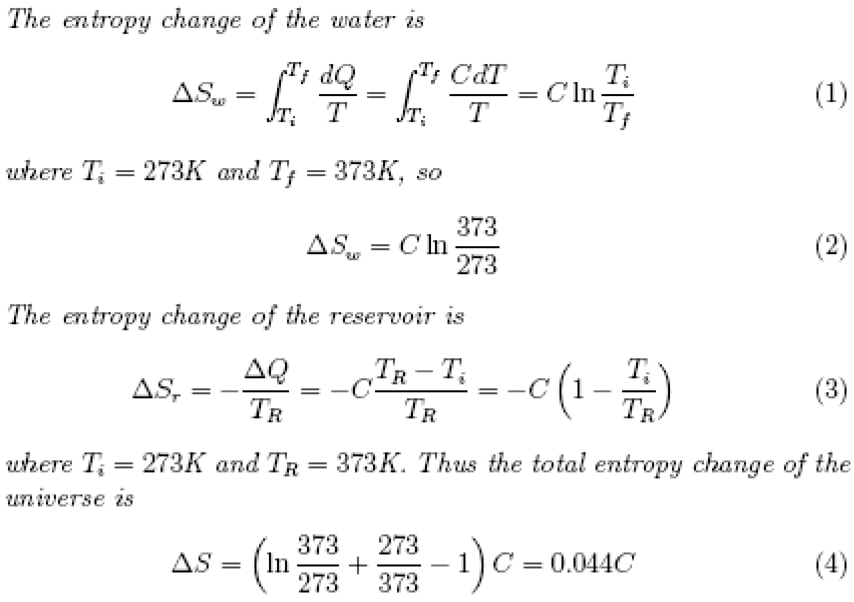
Where, again, the integral just gives a number. We can now calculate:

$$S = -\frac{d\theta}{dT} \sim T^{4/3} \rightarrow \beta = 4/3$$

$$P = -\frac{d\theta}{dA} \sim T^{7/3} \rightarrow \gamma = 7/3$$

Solution 3

a.)



The entropy change of the water is

$$\Delta S_w = \int_{T_i}^{T_f} \frac{dQ}{T} = \int_{T_i}^{T_f} \frac{CdT}{T} = C\ln\frac{T_i}{T_f} \qquad (1)$$

where $T_i = 273K$ and $T_f = 373K$, so

$$\Delta S_w = C\ln\frac{373}{273} \qquad (2)$$

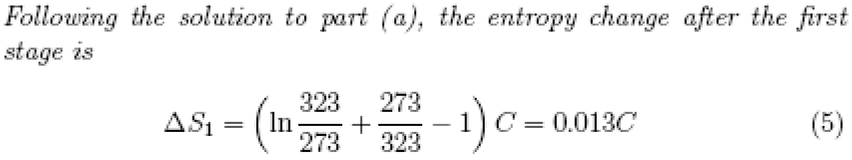
The entropy change of the reservoir is

$$\Delta S_r = -\frac{\Delta Q}{T_R} = -C\frac{T_R - T_i}{T_R} = -C\left(1 - \frac{T_i}{T_R}\right) \qquad (3)$$

where $T_i = 273K$ and $T_R = 373K$. Thus the total entropy change of the universe is

$$\Delta S = \left(\ln\frac{373}{273} + \frac{273}{373} - 1\right)C = 0.044C \qquad (4)$$

b.)



Following the solution to part (a), the entropy change after the first stage is

$$\Delta S_1 = \left(\ln\frac{323}{273} + \frac{273}{323} - 1\right)C = 0.013C \qquad (5)$$

and after the second stage

$$\Delta S_2 = \left(\ln\frac{373}{323} + \frac{323}{373} - 1\right) C = 0.010C \tag{6}$$

Thus the total entropy change of the universe is

$$\Delta S = \Delta S_1 + \Delta S_2 = 0.023C \tag{7}$$

which is less than that found in part (a) (as we should expect).

c.) *In this limit all heat transfer would be adiabatic and* $\Delta S = 0$.

Solution 4

(a) Partition function: $Z = \sum_s e^{-\beta\varepsilon} = e^{-\beta\varepsilon} + e^{-\beta(\varepsilon+\Delta)}$

Free energy: F = -kT lnZ $F = -kT\ln(e^{-\beta\varepsilon} + e^{-\beta(\varepsilon+\Delta)})$

Where β = 1/kT

(b) To calculate the specific heat, you first need the energy:

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{\varepsilon e^{-\beta\varepsilon} + (\varepsilon+\Delta)e^{-\beta(\varepsilon+\Delta)}}{e^{-\beta\varepsilon} + e^{-\beta(\varepsilon+\Delta)}} = \varepsilon + \frac{\Delta}{e^{\Delta/kT} + 1}$$

$$C(T) = \frac{\partial \varepsilon}{\partial T} = \frac{\Delta^2 e^{-\Delta/kT}}{kT^2(1+e^{-\Delta/kT})^2}$$