Physics Qualifying Examination

Problems 1–6	Thursday, January 8, 2009	1–5 pm
Problems 7-12	Friday, January 9, 2009	1-5 pm

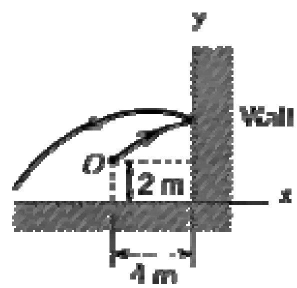
1. Solve each problem.

2. Start each problem solution on a fresh page. You may use multiple pages per problem.

3. At the top of each solution page put the problem number (1–12) and your Social Security number, but not your name or any other information.

Problem 1

A ball is thrown against a wall, making a perfectly elastic collision, and bounces over the head of the thrower as shown in the figure. When it leaves the thrower's hand, the ball is 2 m above the ground and 4 m from the wall, and has $V_x(0)=V_y(0)=10$ m/sec. How far behind the thrower does the ball hit the ground? (Numerical answer required, let $g=10$ m/sec^2.)



Solution

With or without the wall the total time in flight is controlled by the vertical component of velocity and initial position, the coordinate y as a function of time is

$$y = h_0 + V_y(0) - \frac{gt^2}{2} \qquad t = \frac{V_y(0) + \sqrt{V_y^2(0) + 2gh}}{g}$$

Note that reflection from the wall has no effect on the vertical motion. From above t=2.18 sec. During this time the ball travels 21.8 m horizontally. Out of that 4 m to reach the wall and 4 m back to the point above the thrower and then another 13.8 m before hitting the ground.

Problem 2

A pion of mass m_π decays into an electron and an antineutrino with masses m_e, and m_ν. Find the velocity of the antineutrino in the rest frame of the electron in terms of the masses of the three particles. Interpret your answer in the case where $m_\nu = 0$.

Solution

Considering this problem in the rest frame of the electron we have energy-momentum conservation for the decay as follows:

$$(E_\pi, \vec{p}_\pi) = (m_e, 0) + (E_\nu, \vec{p}_\nu)$$

Here we assume units where $c = 1$. By taking a relativistic square of both sides we find

$$m_\pi^2 = m_e^2 + m_\nu^2 + 2E_\nu m_e$$

and thus

$$E_\nu = \frac{m_\pi^2 - m_e^2 - m_\nu^2}{2m_e}.$$

Since $E = \frac{m}{\sqrt{1-v^2}}$ we obtain the final answer

$$v = \sqrt{1 - \left(\frac{2m_e m_\nu}{m_\pi^2 - m_e^2 - m_\nu^2}\right)^2}.$$

For $m_\nu = 0$ we have $v = 1$ which means it is equal to the speed of light.

Problem 3

A particle of mass *m* and energy *E* approaches a one-dimensional step potential of height V_0, i.e. the potential is zero for $x<0$ and $V_0>0$ for $x>0$. Assume the particle is incident from the left.

(a) For $E<V_0$ determine the penetration depth, which is defined as the distance from $x=0$ where the probability to find the particle has dropped to $1/e$ of its value at $x=0$.

(b) For $E>V_0$ determine the reflection probability.

(c) For $E>V_0$ determine the transmission probability.

(d) For the case when *E* is much larger than V_0, find the leading approximation to the way in which the reflection probability varies with increasing energy.

(e) Suppose now that $V_0<0$, so that the step potential is a step downward, and $E>>|V_0|$. Is the reflection coefficient nonzero in this case? Explain! Is there a reflection classically?

Solution

(a) For $x>0$ the solution of the Schrödinger eqn. for $E<V_0$ is $\sim \exp[-\kappa x]$, $\kappa = \sqrt{2m(V_0-E)}/\hbar$

penetration depth : $\frac{1}{2\kappa} = \frac{\hbar}{2\sqrt{2m(V_0-E)}}$

(b) + (c) For $E>V_0$

$x<0$ $\quad \psi(x) = Ae^{ikx} + Be^{-ikx}$ $\quad k = \sqrt{2mE}/\hbar$

$x>0$ $\quad \psi(x) = Ce^{i\alpha x}$ $\quad \alpha = \sqrt{2m(E-V_0)}/\hbar$

continuity of ψ : $A+B=C$

continuity of ψ' : $k(A-B) = \alpha C$

solve for $\frac{B}{A}$ and $\frac{C}{A}$:

$$\frac{B}{A} = \frac{k-\alpha}{k+\alpha} \quad , \quad \frac{C}{A} = \frac{2k}{k+\alpha}$$

Reflection probability : $R = \left|\frac{B}{A}\right|^2 = \left|\frac{k-\alpha}{k+\alpha}\right|^2$

transmission probability : $T = \frac{\hbar\alpha}{\hbar k}\left|\frac{C}{A}\right|^2 = \frac{\alpha}{k}\frac{4k^2}{(k+\alpha)^2} = \frac{4\alpha k}{(k+\alpha)^2}$

verify : $T+R=1$ ✓

(d) $E \gg V_0$

$$R = \left(\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2 \simeq \left(\frac{1-1+\frac{1}{2}\frac{V_0}{E}}{1+1-\frac{1}{2}\frac{V_0}{E}}\right)^2 \simeq \frac{1}{16}\left(\frac{V_0}{E}\right)^2$$

$R \to 0$ as $\left(\frac{V_0}{E}\right)^2 \to 0$

(e) $R = \left(\frac{\sqrt{E}-\sqrt{E+V_0}}{\sqrt{E}+\sqrt{E+V_0}}\right)^2 \neq 0$ $\qquad R \to \left(\frac{V_0}{4E}\right)^2$

contrary to classical expectation!

Problem 4

Consider a *2*-level system with energy states ε and $\varepsilon + \Delta$ $(\Delta \geq 0)$.
(a) Compute the partition function and the free energy.
(b) Derive an expression for the specific heat $C(T)$.
(c) What are the low-T and high-T asymptotic forms of the above expression? Sketch your result.

Solution:
(a) Partition function: $Z = \sum_{\{E\}} e^{-\beta E} = e^{-\beta\varepsilon} + e^{-\beta(\varepsilon+\Delta)}$.

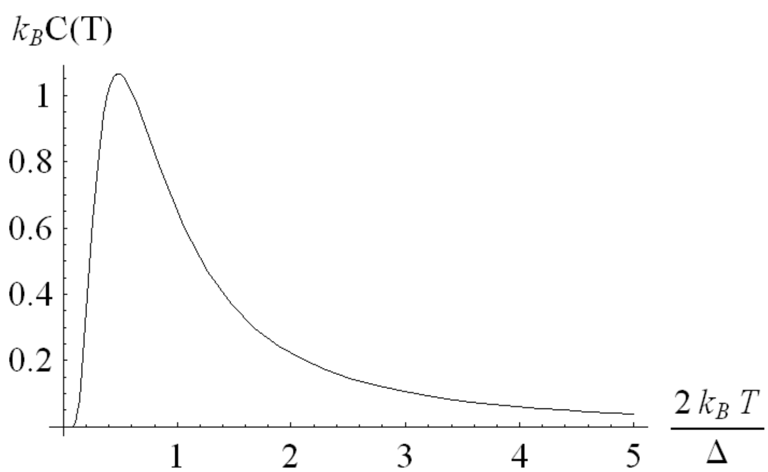
Free energy: $F = -k_B T \ln(Z) = -k_B T \ln(e^{-\beta\varepsilon} + e^{-\beta(\varepsilon+\Delta)})$, where $\beta = \dfrac{1}{k_B T}$.

(b) The specific heat can be obtained from the Free energy:

$$C(T) = -T\frac{\partial^2 F}{\partial T^2} = \frac{\Delta^2}{4k_B T^2}\frac{1}{\cosh^2\left(\dfrac{\Delta}{2k_B T}\right)}.$$

(c) At low T: $C(T) \to \dfrac{\Delta^2}{k_B T^2} e^{-\frac{\Delta}{k_B T}}$.

At high T : $C(T) \to \dfrac{\Delta^2}{4k_B T^2}$.



Problem 5

Consider a spinless non-relativistic quantum particle of charge q and mass m moving in 3D in a homogeneous magnetic field $\vec{B} = B\hat{z}$ *and* subject to a parabolic potential $V(x, y, z) = \frac{1}{2}\kappa x^2$.

(a) Find a vector potential $\vec{A}$ in a gauge where $\vec{A}$ points in the $\hat{y}$ direction
(b) Write down the Hamiltonian operator for the above problem.
(c) Find the energy spectrum.

Solutions:
(a) $\vec{A} = Bx\hat{y}$.
(b) $H = \frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{q}{c}Bx\right)^2 + \frac{p_z^2}{2m} + \frac{1}{2}\kappa x^2$.
(c) $E(n, k_y, k_z) = \hbar\sqrt{\frac{\kappa}{m} + \left(\frac{qB}{mc}\right)^2}\left(n + \frac{1}{2}\right) + \frac{\hbar^2 k_y^2}{2m}\left(\frac{c\kappa}{qB + c\kappa}\right) + \frac{\hbar^2 k_z^2}{2m}; n = 0,1,2...$

Problem 6

Consider a quantum mechanical system with three possible orthonormal states ("colors" red, blue and green). Any wave function $|\psi\rangle$ may be written as a linear combination of the three basis states $|R\rangle$, $|B\rangle$, and $|G\rangle$. The system is described by the Hamiltonian

$$H = E_0\,(2|R\rangle\langle R| + 2|B\rangle\langle B| + 2|G\rangle\langle G| - |G\rangle\langle B| - |B\rangle\langle G|)$$

1. Use the vector basis $|R\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, |B\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |G\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$, express the Hamiltonian as a matrix, and find the normalized energy eigenstates and their corresponding eigenvalues.

2. Assume that at time $t=0$ the wave function is $|\psi(0)\rangle = |G\rangle$. Find the wave function $|\psi(t)\rangle$ at an arbitrary time *t*. What are the respective probabilities for the color to be measured as red, green, or blue at time *t* ?

Solution:

1. (6 points)

$$H = E_0 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

Eigenvalues: Find λ for $|H - \lambda I| = 0$.

$$\begin{aligned} (2E_0 - \lambda)^3 - (2E_0 - \lambda)\,E_0^2 &= 0 \\ (2E_0 - \lambda)\left[(2E_0 - \lambda)^2 - E_0^2\right] &= 0 \\ (2E_0 - \lambda)\,(E_0 - \lambda)\,(3E_0 - \lambda) &= 0 \end{aligned}$$

Eigenvalues and corresponding eigenfunctions: $(E_1 = 2E_0,\ \psi_1)$, $(E_2 = E_0,\ \psi_2)$, and $(E_3 = 3E_0,\ \psi_3)$ with (phases are chosen real)

$$\psi_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} = |R\rangle\,, \quad \psi_{\substack{2\\3}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm 1\end{pmatrix} = \frac{1}{\sqrt{2}}\,(|B\rangle \pm |G\rangle)\ .$$

2. (4 points) For $i = 1, 2, 3$ the time evolution of the eigenstates is

$$\psi_i(t) = \psi_i(0)\, e^{-i\omega_i} \;\text{ with }\; \omega_i = E_i/\hbar \;.$$

For $\psi(0) = |G\rangle = 2^{-1/2}\, (\psi_2 - \psi_3)$ and $\omega_0 = E_0/\hbar$ we get

$$\begin{aligned} \psi(t) &= \frac{1}{\sqrt{2}} \left(\psi_2\, e^{-i\omega_0 t} - \psi_3\, e^{-3i\omega_0 t} \right) \\ &= \frac{1}{2} e^{-2i\omega_0 t} \left[(\psi_B + \psi_G)\, e^{i\omega_0 t} - (\psi_B - \psi_G)\, e^{-i\omega_0 t} \right] \\ &= e^{-2i\omega_0 t} \left[\psi_G \cos(\omega_0 t) + i\psi_B \sin(\omega_0 t) \right] \end{aligned}$$

The probabilities are $P_X = |\langle X|\psi(t)\rangle|^2$ with $X = R,\, B,\, G$. Therefore,

$$P_R = 0,\;\; P_G = \cos^2(\omega_0 t), \;\;\text{ and }\;\; P_B = \sin^2(\omega_0 t) \;.$$

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Problem 7

For each of the following energy quantities, give an expression in terms of fundamental constants which is dimensionally correct (numerical prefactors are not required) and compute from your expression an order of magnitude estimate in units of electron Volt (eV).

Useful information (formulas in Gaussian cgs units):

• Rydberg: $1\,\text{Ry} = m_e e^4 / (2\hbar^2) \approx 13.6\,\text{eV}$.

• Bohr Magneton: $\mu_B = e\hbar / (2m_e c) \approx 6\times10^{-5}$ eV/T (T Tesla).

• Fine Structure Constant: $\alpha = e^2 / (\hbar c) \approx 1/137$.

Example: The electron kinetic energy in the ground state of the Hydrogen atom.

Solution: Denote the size of the atom by a_B . From Newton's second law one finds $T = m_e v^2 / 2 = e^2 / (2a_B)$ for the kinetic energy and the quantization condition $m_e v a_B = \hbar$ leads to $T \sim \hbar^2 / (m_e a_B^2)$. When equated with the potential energy e^2 / a_B , this yields $a_B = \hbar^2 / (m_e e^2) \approx 5\times10^{-11}\text{m}$. The kinetic energy is then $T \sim m_e e^4 / \hbar^2 = 2\ \text{Ry} \approx 10\ \text{eV}$. (One Ry is 13.6 eV, but the order of magnitude 10 eV is sufficient for this problem).

1. The ground state binding energy of the electron in a U^{91-} ion (a Uranium ion with atomic number 92, atomic weight 238, which is 91–times ionized so that there is only one electron present).

2. The ground state energy level splitting in a Hydrogen atom due to the spin–orbit interaction (fine structure splitting).

3. The hyperfine splitting in a Hydrogen atom (the splitting of electronic energy levels due to the interaction with the magnetic moment of the proton).

4. The relativistic correction to the binding energy of a Hydrogen atom.

5. The Zeeman splitting of electron levels in a Hydrogen atom when the magnetic field is 1 T (T Tesla).

6. The energy of a nucleon in a typical nucleus.

7. The rotational energy of an H_2 molecule.

Solutions:

1. Using the scaling $e^2 \to Ze^2$ with $Z = 92$ in the expression for the Hydrogen atom, we obtain

$$E \sim \frac{mZ^2e^4}{\hbar^2} \approx 10^4\ \mathrm{Ry} \approx 10^5\ \mathrm{eV}\ .$$

2. This is the energy of the electron spin dipole in the magnetic field due to its orbital motion (E_r radial electric field, p momentum)

$$\triangle E \sim \mu_B \frac{vE_r}{c} \sim \frac{e\hbar}{m_e c} \frac{pe}{m_e c a_B^2} \sim \frac{e^8 m_e}{\hbar^4 c^2} \sim \alpha^2\,\mathrm{Ry} \approx 10^{-3}\ \mathrm{eV}.$$

3. It is reduced by a factor of the mass ratio from the fine structure splitting

$$\triangle E \sim \frac{m_e}{m_P} \alpha^2\ \mathrm{Ry} \approx 10^{-6}\ \mathrm{eV}.$$

4. By Taylor expansion of $\sqrt{m_e^2c^4 + p^2c^2}$

$$\triangle E \sim m_e c^2 \left(\frac{p^2}{m_e^2c^2}\right)^2 \sim \alpha^2\,\mathrm{Ry} \approx 10^{-3}\ \mathrm{eV}.$$

5. The Zeeman energy in the magnetic field B is

$$\triangle E \sim \mu_B B \approx\ 10^{-4}\ \mathrm{eV}.$$

6. In a nucleus of radius $r_N \approx 1$ fm $= 10^{-15}$ m (T kinetic energy):

$$E_N \sim T_N = \frac{\hbar^2}{m_N r_N^2} \approx \frac{T_N}{T_e}\,10\,\mathrm{eV} = \frac{m_e}{m_N}\frac{r_N^2}{a_B^2}\,10\,\mathrm{eV} \approx 10^7\ \mathrm{eV}.$$

7. The rotational energy is (L angular momentum, μ reduced mass)

$$E_{\mathrm{rot}} \sim \frac{L^2}{\mu r^2} \sim \frac{\hbar^2}{m_H a_B^2} \sim \frac{m_e}{m_H}\ \mathrm{Ry} \approx 10^{-2}\ \mathrm{eV}.$$

Problem 8

(a) Consider a photon gas at temperature T = 2.9 K. Using $E = uV$ and $dE = -PdV$ for an adiabatic, quasistatic process, derive an expression for how the energy density u depends upon the volume V for an adiabatic, quasi-static expansion of the gas. Hint: $P = u/3$ for a photon gas.

(b) For a photon gas, $u \sim T^4$. Using this, derive an expression for how the volume V depends upon the temperature T for an adiabatic, quasistatic process.

(c) If the universe expanded quasi-statically and adiabatically from an initial state where the radiation temperature was T = 3000 K to the current state with a temperature of T = 2.9 K, what is the ratio of the current volume to the initial volume? Assume that the universe is composed of no matter.

Solution:

(a) $dE = -P\,dV$

$E = uV \rightarrow dE = udV + Vdu = -PdV = -(u/3)dV$

Therefore: $-(4/3)udV = Vdu$, or $(du/u) = -4/3\,(dV/V)$

This results in: $uV^{4/3} = \text{constant}$, or $u \sim V^{-4/3}$

(b) Using what we have above, $T^4V^{4/3} = \text{constant}$, or $V \sim T^{-3}$

(c) $(V_{final}/V_{initial}) = (T_{initial}/T_{final})^3 = (3000/2.9)^3 = 1.1 \times 10^9$

Problem 9

Estimate each of the following, with an explanation for your reasoning:

(a) the average kinetic energy of a monatomic air molecule in this room.

(b) the molar heat capacity at constant volume of a rock at room temperature.

(c) the electrical resistance of a metal wire one meter long and 1 mm in radius. (Hint: $\rho \sim 10^{-8}\Omega m$).

(d) the number of atoms in 1 cm^2 on the surface of a solid metal.

(e) the number of atoms in 1 m^3 of gas at room temperature and atmospheric pressure.

Solution:

(a) 3/2 kT = 6.2×10^{-21} J (for T ~ 300 K)

(b) Cv = 3R = 24.9 J/mol K

(c) R = ρ(L/A) = $10^{-8}/(\pi * 10^{-6}) = 0.3 \times 10^{-2} \sim 3 \times 10^{-3}$ Ohm

(d) Interatomic spacing of solids is ~ 4×10^{-10} m. Each atom then occupies an area on a surface of roughly 1.9×10^{-19} m^2. For 1 cm^2, which has an area of 1×10^{-4} m^2, we then have: number of atoms = 1×10^{-4} m^2/1.6×10^{-19} m^2 = 6.25×10^{18} atoms

(e) PV = NkT -> N = PV/kT = $(101 \times 10^3$ Pa$)(1$ $m^3)/(1.38 \times 10^{-23} * 300) = 2.4 \times 10^{25}$

Problem 10

A positive charge is uniformly distributed throughout a very long cylindrical volume of radius R. The charge per unit volume is ρ.

(a) Find the electric field $\bar{\boldsymbol{E}}$ everywhere as a function of the distance $\boldsymbol{r}$ from the axis of the cylinder.

(b) Find the electric potential V everywhere as a function of $\boldsymbol{r}$. Define $V = 0$ at the surface of the cylinder.

(c) Sketch $\boldsymbol{E}$ and V as function of $\boldsymbol{r}$, from $\boldsymbol{r} = 0$ to $\boldsymbol{r} = 3R$, showing the values of each at $\boldsymbol{r} = 0$, R, and $3R$.

Solution:

(a) Use Gauss's law to find the electric field outside the cylinder. If we choose a Gaussian surface to be a cylinder of length L and radius $r > \text{R}$. Then by symmetry the field will be point radially outward on the curved part of the Gaussian surface and must be parallel to the surface on the end caps. This means that the flux of the electric field through this surface is given by

$$\oint \mathbf{E} \cdot d\mathbf{A} = \text{E}(2\pi rL)$$

But Gauss's law tells us that this is also Q_{encl}/ε_0, where Q_{encl} is the enclosed charge. The volume of the charged cylinder cut out by our surface is $V = \pi R^2 L$, so the enclosed charge is $Q_{encl} = \pi R^2 L\rho$.

Thus Gauss's law gives:

$$E(2\pi rL) = 1/\varepsilon_o \cdot \pi R^2 L\rho \quad \longrightarrow \quad E = \rho R^2/2\varepsilon_o r$$

Since the electric field is radially outward, we have

$$\mathbf{E} = \rho R^2/2\varepsilon_o r \;\; \acute{\mathbf{r}} \qquad \textit{(outside)}$$

Inside the cylinder, the same technique will be used, and the process is the same as before, except now the charge enclosed by the Gaussian surface is reduced to $Q_{encl} = (\pi r^2 L)\rho$ which gives an electric field of

$$\mathbf{E} = \rho r/2\varepsilon_o \;\; \acute{\mathbf{r}} \qquad \textit{(inside)}$$

(b) Start by solving for the potential outside the cylinder, we know that the potential will be the maximum at the surface of the cylinder, because the electric field always point outward (and so a charged particle will lose energy as it moves away from the cylinder). This means that

$$V(r') = -\int_R^{r'} \mathbf{E}(r) \cdot d\mathbf{r} = -\int_R^{r'} \rho R^2/2\varepsilon_o r \; dr$$

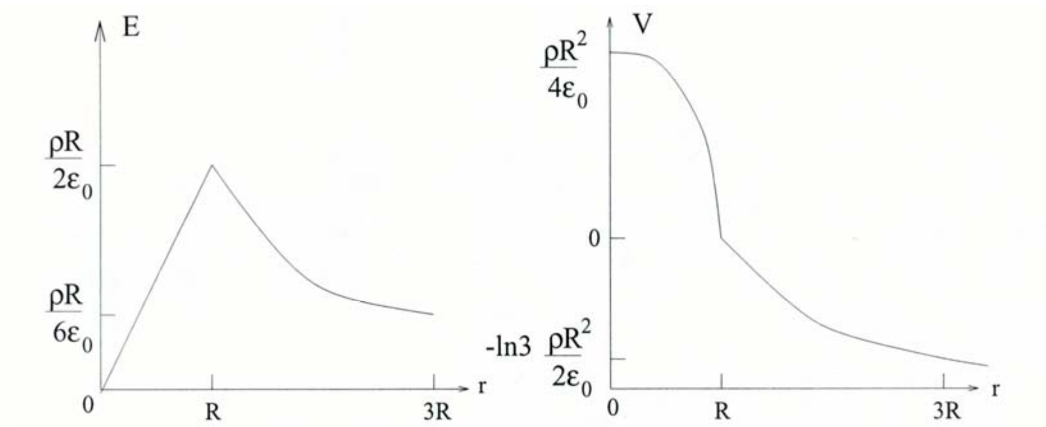
$$= -\rho R^2/2\varepsilon_o \cdot \ln(r'/R) \quad \textit{(outside)}$$

Inside the cylinder the potential increases as we move away from r = R, so we have

$$V(r') = -\int_R^{r'} \mathbf{E}(r) \cdot d\mathbf{r} = -\int_R^{r'} \rho r/2\varepsilon_o \; dr$$

$$= -\rho/4\varepsilon_o \cdot (R^2 - r'^2) \quad \textit{(inside)}$$

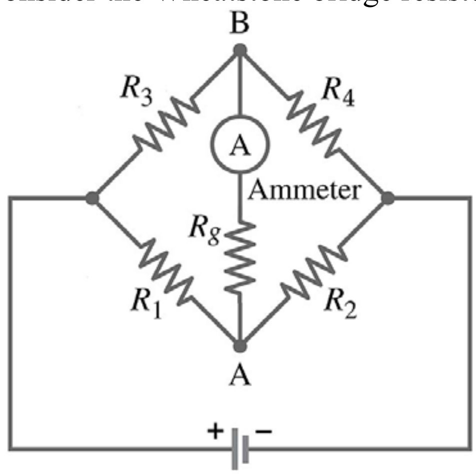
(c)



Note that the potential falls without bound as we move away from the wire, essentially because the wire has an infinite charge.

Problem 11

Consider the Wheatstone bridge resistor circuit shown in the diagram below.



(a) Use Kirchhoff's laws to derive a condition for the values of R_1, R_2, R_3, and R_4, under which the galvanometer does not measure a current and the Wheatstone bridge is "balanced".

(b) What is the total resistance of the circuit? The total resistance is defined as the ratio of the battery voltage over the current through the galvanometer.

Solution:

(a) We will start by labeling I_i as the current in resistor i, assuming each goes tothe right except the galvanometer current I_g, which we can assume goes downward. Thus we have five unknown currents. We can apply Kirchoff's laws to the circuit, which give that the current going into a branch in the circuit is the sum of the currents in the two branches (conservation of charge), and that the voltage drops around any loop in the circuit add to zero. The first (conservation of charge) gives

$$I_1 = I_2 + I_g \quad (1)$$
$$I_3 + I_g = I_4 \quad (2)$$

The second criterion (sum of the voltage drops around a loop equals zero) can then be applied to the three loops in the circuit; the first is the loop made up by the three resistors on the left, which (starting at Aand add voltage rises counterclockwise) gives:

$$-I_gR_g + I_3R_3 - I_1R_1 = 0 \quad (3)$$

Similarly for the loop of resistors on the right (start at A and add voltage rises counterclockwise)

$$-I_2R_2 + I_4R_4 + I_gR_g = 0 \quad (4)$$

We have one more equation which is for a loop containing the battery, and thus ε; note we can choose any of a number of loops and because we have already written down the equations for the interior loops they will all amount to the same thing, so choose the bottom branch of the circuit and start at the - side of the emf,

$$\varepsilon - I_1R_1 - I_2R_2 = 0 \qquad (5)$$

We now have five equations in five unknowns which we can solve, however tedious that process is, for the five currents. It is better to do this without substituting for the resistors so that we have the general formula for the galvanometer current.

Before starting on what is formally the job of inverting a 5x5 matrix, it helps to get organized. Equation *(1)* can be used to eliminate I_1 from every equation where it appears. The same is true of Eq. *(2)* and I_4. Once we have used these we have only three equations left. Our new system of equations looks like

$$-I_gR_g + I_3R_3 - I_2R_1 - I_gR_1 = -I_g(R_g + R_1) - I_2R_1 + I_3R_3 = 0 \qquad (6)$$
$$-I_2R_2 + I_3R_4 + I_gR_4 + I_gR_g = I_g(R_g + R_4) - I_2R_2 + I_3R_4 = 0 \qquad (7)$$
$$\varepsilon - I_2R_1 - I_gR_1 - I_2R_2 = \varepsilon - I_gR_1 - I_2(R_1 + R_2) = 0 \qquad (8)$$

where it obviously pays to collect terms and organize the coeficients in terms of the unknown currents. The best way to proceed now is to combine *(6)* and *(7)* to eliminate I_3 which does not appear in *(8),* i.e.

$$R_4 \text{ x } (6): \quad -I_g[(R_g + R_1)R_4] - I_2R_1R_4 + I_3R_3R_4 = 0 \qquad (9)$$
$$R_3 \text{ x } (7): \quad I_g[(R_g + R_4)R_3] - I_2R_2R_3 + I_3R_3R_4 = 0 \qquad (10)$$
$$(10) - (9): \quad I_g[R_g(R_3 + R_4) + R_4(R_1 + R_3)] - I_2[R_2R_3 - R_1R_4)] = 0: \qquad (11)$$

Substituting the value of I_2 in terms of I_g from *(8),* we have

$$I_g\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] - \left[(\varepsilon - I_gR_1)/(R_1 + R_2)\right]\left[R_2R_3 - R_1R_4)\right] = 0; \qquad (12)$$

or that

$$I_g\left\{(R_1 + R_2)\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] + R_1(R_2R_3 - R_1R_4)\right\} = \varepsilon\left[R_2R_3 - R_1R_4\right]; \qquad (13)$$

so that, finally,

$$I_g = \varepsilon\left[R_2R_3 - R_1R_4\right] / \left\{(R_1 + R_2)\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] + R_1(R_2R_3 - R_1R_4)\right\} \qquad (14)$$

We were asked to find the condition where the Wheatstone bridge is "balanced". That occurs when I_g goes to zero, which happens only if the numerator of the equation *(14)* goes to zero. That is, the bridge balances only if:

$$R_2R_3 - R_1R_4 = 0$$

(Another way of seeing this is to note that the current through the cross branch of the Wheatstone bridge goes to zero, when the voltages at points A and B are equal). We can obtain this condition by breaking the galvanometer circuit (removing the cross branch) and obtaining the voltages at points A and B, which are:

$$V_A = \varepsilon\left[1 - R_1/(R_1 + R_2)\right]$$
$$V_B = \varepsilon\left[1 - R_3/(R_3 + R_4)\right]$$

And if we set these two equal:

$$\left[1 - R_1/(R_1 + R_2)\right] = \left[1 - R_3/(R_3 + R_4)\right]$$
$$R_1/(R_1 + R_2) = R_3/(R_3 + R_4)$$
$$R_1(R_3 + R_4) = R_3(R_1 + R_2)$$
$$R_1R_3 + R_1R_4 = R_3R_1 + R_3R_2$$

$$R_1R_4 = R_3R_2$$

Which is that same condition we came up with above using Kirchoff's laws.

(b) This result was obtained in the process of solving Kirchoff's laws above. Starting with equation *(14)*., and noting that $V = IR \rightarrow R = \varepsilon/I$

$$I_g = \varepsilon\left[R_2R_3 - R_1R_4\right] / \left\{(R_1 + R_2)\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] + R_1(R_2R_3 - R_1R_4)\right\} \quad (14)$$

Thus

$$R_{\text{total}} = \left\{(R_1 + R_2)\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] + R_1(R_2R_3 - R_1R_4)\right\} . / \left[R_2R_3 - R_1R_4\right]$$

It can be noted that the solution to the balance condition can be obtained from the insight that $V_A = V_B$, but the equivalent resistance can ONLY be obtained from the derivation using Kirchoff's laws.

Problem 12

A very flexible uniform chain of length L and linear mass density σ is suspended from one end so that it hangs vertically, with the lower end just touching the surface of a scale. The upper end is suddenly released so that the chain falls onto the scale and coils up in a small heap, each link coming to rest the instant it strikes the scale. Calculate the force measured by the scale as a function of the distance x that the top of the chain has fallen in the following steps:

1) Find the velocity of the chain in terms of the distance x.

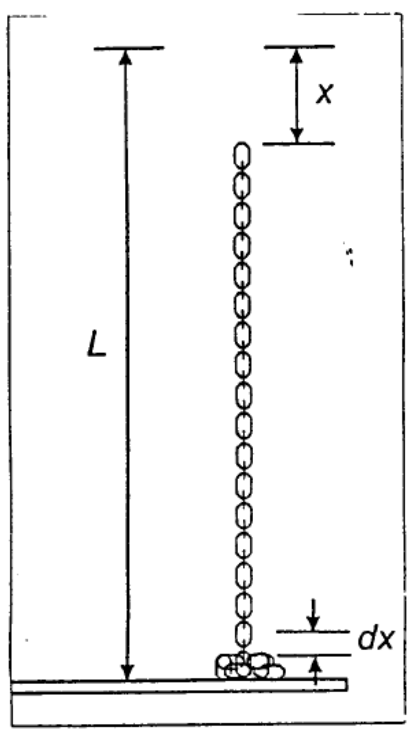
2) What is the force due to the momentum change as a function of x?

3) What is the force due to the weight as a function of x?

4) What is the total force in terms of x?

5) What is the total force the instant before $x = L$?

6) What is the final total force after the chain has completely fallen?



Chain, of total length L, has fallen a distance x. Length element dx (of mass dm) is about to be stopped by striking the table.

Solution:

1) Find the velocity of the chain in terms of the distance $\boldsymbol{x}$.

$$v(x) = (2gx)^{1/2}$$

2) What is the force due to the momentum change as a function of $\boldsymbol{x}$?

$$F_m = dp/dt = v\ dm/dt = v\ d(\sigma x)/dt = v^2\sigma = 2g\sigma x$$

3) What is the force due to the weight as a function of $\boldsymbol{x}$?

$$F_W = g\sigma x$$

4) What is the total force in terms of $\boldsymbol{x}$?

$$F_T = 3g\sigma x$$

5) What is the total force the instant before $\boldsymbol{x} = \boldsymbol{L}$?

$$F_T = 3g\sigma L$$

6) What is the final total force after the chain has completely fallen?

$$F_T = g\sigma L$$