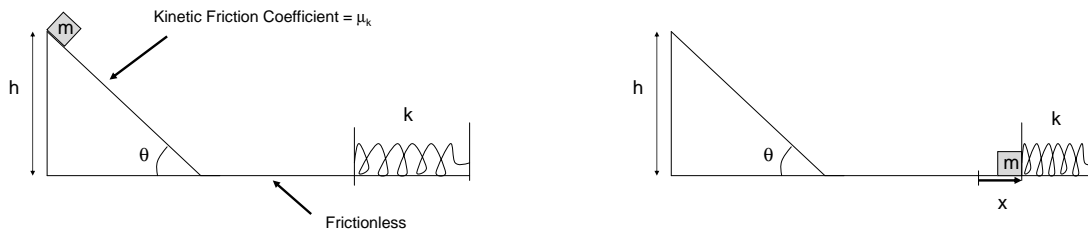


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1. (10 pts) A block of mass  $m$  is released from rest on an incline which makes an angle  $\theta$  with the horizontal (see figure). The block slides down the surface of the incline, which has a coefficient of kinetic friction  $\mu_k$ , and then slides on a *frictionless* level surface hitting a spring of spring constant  $k$ . The initial height of the block above the level surface is  $h$ , and the maximum compression of the spring is  $x$ .



- (a) Find an expression for  $\mu_k$ , the coefficient of kinetic friction of the incline. Express your answer in terms of  $m$ ,  $k$ ,  $g$ ,  $h$ ,  $x$  and  $\theta$ .

*On the incline, the frictional force is  $f = \mu_k N = \mu_k mg \cos \theta$ , and the distance the block slides is  $d = h / \sin \theta$ . The work-energy theorem then tells us that the initial energy ( $mgh$ ), plus the (negative) work done by friction ( $-fd$ ), is equal to the final energy ( $\frac{1}{2}kx^2$ ):*

$$mgh - fd = \frac{1}{2}kx^2$$

$$mgh - \mu_k mg \cos \theta \frac{h}{\sin \theta} = \frac{1}{2}kx^2$$

*Solving for  $\mu_k$  gives*

$$\Rightarrow \mu_k = \tan \theta \left( 1 - \frac{kx^2}{2mgh} \right)$$

- (b) What is the velocity of the block as it slides on the level surface?

$$\frac{1}{2}mv^2 = \frac{1}{2}kx^2 \Rightarrow v = \sqrt{\frac{k}{m}}x$$

2. (10 pts) A particle of mass  $m$  moves horizontally and feels a damping force which is proportional to its velocity,  $F_r = -mkv$ . The initial conditions for the particle are  $x(t=0) = 0$  and  $v(t=0) = v_0$ .

(a) Write the equation of motion for this particle and solve it to obtain the velocity as a function of time,  $v(t)$ .

$$m \frac{dv}{dt} = -mkv \Rightarrow dt = -\frac{1}{k} \frac{dv}{v}$$

$$t = -\frac{1}{k} \int_{v_0}^v \frac{dv}{v} = -\frac{1}{k} \ln \frac{v}{v_0}$$

$$\Rightarrow v(t) = v_0 e^{-kt}$$

(b) Integrate  $v(t)$  to obtain the position as a function of time,  $x(t)$ .

$$x(t) = \int_0^t v(t) dt = \int_0^t v_0 e^{-kt} dt = -\frac{v_0}{k} e^{-kt} \Big|_0^t$$

$$\Rightarrow x(t) = \frac{v_0}{k} (1 - e^{-kt})$$

- (c) Write a differential equation for the velocity as a function of position,  $v(x)$ , using the fact that  $\frac{dv}{dt} = v \frac{dv}{dx}$ , and solve it to obtain  $v(x)$ .

$$m \frac{dv}{dt} = mv \frac{dv}{dx} = -mkv \Rightarrow dv = -kdx$$

$$\int_{v_0}^v dv = -k \int_0^x dx \Rightarrow v - v_0 = -kx$$

$$\Rightarrow v(x) = v_0 - kx$$

- (d) What is the displacement of the particle in the limit of long times?

*In the limit  $t \rightarrow \infty$  the velocity  $v$  goes to zero. It follows from (c) that the displacement goes to,*

$$x \rightarrow \frac{v_0}{k}$$

3. (10 pts) An oscillator consists of a block of mass  $m = 0.2$  kg which is attached to a spring with spring constant  $k = 3.2$  N/m, and which feels a damping force  $F_r = -b\dot{x}$ , where  $x(t)$  is the displacement of the block at time  $t$ .

(a) For what value of the damping coefficient  $b$  will this oscillator be *critically damped*?

*For the oscillator to be critically damped, we require*

$$\beta = \frac{b}{2m} = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.2 \text{ N/m}}{0.2 \text{ kg}}} = 4 \text{ s}^{-1}$$

$$\Rightarrow b = 2m\beta = 2(0.2 \text{ kg})(4 \text{ s}^{-1}) = 1.6 \text{ kg/s}$$

(b) When  $b$  has the value found in Part (a), determine  $x(t)$  using the initial conditions  $\dot{x}(t=0) = 2$  m/s and  $x(t=0) = 0$ .

*For a critically damped oscillator,*

$$x(t) = (A + Bt)e^{-\beta t}; \quad \dot{x}(t) = (B - \beta A - \beta Bt)e^{-\beta t}$$

$$x(0) = A = 0; \quad \dot{x}(0) = B - \beta A = v_0 \Rightarrow B = v_0$$

$$\Rightarrow x(t) = v_0 t e^{-\beta t} = (2 \text{ m/s}) t e^{-(4 \text{ s}^{-1})t}$$

(c) For the initial conditions from Part (b), what is the maximum displacement reached by the block? (Hint: Use the fact that when the displacement is maximum the velocity is zero.)

*From (b) we have*

$$\dot{x}(t) = (B - \beta A - \beta Bt)e^{-\beta t} \Rightarrow \dot{x}(t) = v_0(1 - \beta t)e^{-\beta t}$$

*Setting this to zero and solving for  $t$  gives,*

$$\dot{x}(t) = 0 \Rightarrow 1 - \beta t = 0 \Rightarrow t = \frac{1}{\beta}$$

*and so the maximum displacement is*

$$x_{max} = x(t = 1/\beta) = \frac{v_0}{\beta} e^{-\beta(1/\beta)} = \frac{v_0}{\beta} e^{-1}$$

$$\Rightarrow x_{max} = \frac{2 \text{ m/s}}{4 \text{ s}^{-1}} e^{-1} = (0.5 \text{ m}) e^{-1} = .184 \text{ m}$$