QUANTUM MECHANICS

Homework set #17:

Time-dependent perturbation and symmetries

Problem # 17.1 : Electron on He surface

An electron close to the surface of liquid helium experiences an attractive force due to the electrostatic polarization of the helium and a repulsive force due to the exclusion principle (hard core). To a reasonable approximation

$$V(z) = \begin{cases} \infty & \text{for } z \leq 0 \\ -Q^2e^2/z & \text{for } z > 0 \end{cases}$$

describes the potential, where $Q^2 = (\epsilon - 1)/[4(\epsilon + 1)]$ and $\epsilon \approx 1.0004$ is the dielectric constant of helium. Here we have assumed that the helium fills the space $z < 0$.

(a) Write the 3-dimensional Schrödinger equation in Cartesian coordinates. Using a separation of variables Ansatz obtain the differential equation for the z-direction and compare it with the radial Schrödinger equation for the hydrogen atom.

(b) Give the eigenvalues and eigenfunctions (in terms of H-atom eigenfunctions) of the electron near the helium surface.

(c) Give numerical values for the binding energy in electron volts and the expectation value of z in Angstroms for an electron in the ground state.

(d) A time dependent electric field produces the following perturbation

$$H' = \begin{cases} eEze^{-t/\tau} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} .$$

Assume that the electron is initially in its ground state and that first order time-dependent perturbation theory is valid. Find an expression for the probability that the electron is in its first excited state for times t much greater than τ.

Problem # 17.2 : positronium

Consider the following schematic spin-dependent Hamiltonian for positronium in a constant magnetic field directed along the z-axis, $H = H_0 + V$,

$$H_0 = A\vec{S}_1 \cdot \vec{S}_2 \quad , \quad V = \frac{eB}{mc}(S_{1z} - S_{2z}) \quad .$$

(a) Find the exact eigenvalues and eigenstates of H_0.

(b) Evaluate the matrix elements of V in this basis.

(c) Deduce the eigenvalues and eigenstates of H to lowest non-vanishing order in time-independent perturbation theory.

(d) Assume the system is in its ground state and subjected for $t \geq 0$ to a weak oscillating magnetic field directed along the z-axis,

$$\vec{B}'(t) = \hat{k}B' \cos(\omega t) \ .$$

Using lowest order time-dependent perturbation theory, evaluate the probability of finding the system in its exited states as a function of time. Assume $\vec{B}'(t)$ is turned on at $t = 0$. You don't need to evaluate the time dependent integrals that appear. It is sufficient to use the unperturbed wave-functions of H, i.e. the eigenfunctions of H_0, for this calculation.

Problem # 17.3 : Lattice translation as a discrete symmetry

Consider a particle in a periodic potential in one-dimension,

$$H = \frac{p^2}{2m} + V(x) \quad , \quad V(x \pm a) = V(x) \quad .$$

Here a is the "lattice constant", the smallest translational unit. A translation by an integer multiple of a leaves the Hamiltonian invariant, i.e. if $\tau(a)$ is the operator translating the system by a, then $\tau^\dagger(a) H \tau(a) = H$.

(a) Show that $[H, \tau(a)] = 0$. What properties do the eigenvectors and eigenvalues of $\tau(a)$ have? Is $\tau(a)$ unitary? Is $\tau(a)$ hermitian?

(b) Let $|n\rangle$ be the ket for the particle being localized at site n. Then obviously $\tau(a)|n\rangle = |n+1\rangle$. Form the parametric (of a continuous real parameter θ, $-\pi \leq \theta \leq \pi$) linear combination

$$|\theta\rangle \equiv \sum_{n=-\infty}^{\infty} e^{in\theta}|n\rangle$$

and show that $|\theta\rangle$ is an eigenfunction of $\tau(a)$. What is the corresponding eigenvalue?

(c) Assume that the wave-function $\langle x|n\rangle$ is only large at site n, small at the neighboring sites $n \pm 1$, and zero elsewhere. Hence, only matrix elements of the Hamiltonian, $\langle n'|H|n\rangle$ for $n' = n$ and for $n' = n \pm 1$ are non-zero. Denote the corresponding values with E_0 and $-\Delta$, respectively. Write down the expression for $H|n\rangle$. Is $|n\rangle$ an eigenket of H? Explain why. Show that $|\theta\rangle$ is an eigenfunction of H and find the eigenvalue.

(d) The spread of the energy as θ varies from $-\pi$ to π is called an energy band. What is the band-width for the present example? If the system is enclosed in a box of length Na with periodic boundary conditions, what are the values θ can take?

Problem # 17.4 : Lenz-Runge-Lagrange vector

Consider the hydrogen atom. The vector $\vec{A}$ known as the Lenz-Runge-Laplace vector is defined as

$$\vec{A} = \frac{1}{2m}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \frac{Ze^2}{r}\vec{r}\,.$$

(a) Show that $\vec{A}$ is hermitian, i.e. $\vec{A}^\dagger = \vec{A}$

(b) Show that $\vec{A} \cdot \vec{L} = \vec{L} \cdot \vec{A} = 0$, i.e. $\vec{A}$ is normal to $\vec{L}$.

(c) Show that $\vec{A}$ is a constant of motion, i.e. $[\vec{A}, H] = 0$. Here $H = \vec{p}^2/2m - Ze^2/r$ is the Hamiltonian of the hydrogen atom.