# QUANTUM MECHANICS

## Homework set #18:

Addition of angular momenta

**Problem # 18.1 :** Clebsch-Gordan coefficients

In a $Ce^{3+}$-ion all electronic shells are either completely filled or empty, except for the 4f-shell, which contains one electron. What are the values of the orbital angular momentum $L$ and the spin $S$? What are the possible values of $J$?

$Ce^{3+}$-ions have a strong spin-orbit coupling, so that $\vec{J}^2$ and the $z$-component of $\vec{J}$, $M_J$, are good quantum numbers. Express the eigen-states of $|LSJM_J\rangle$ for the smallest value of $J$ in the basis $|LSm_Lm_S\rangle$. Determine the Clebsch-Gordan coefficients by making use of the recurrence relations and the normalization condition of the coefficients.

**Problem # 18.2 :** Integrals over spherical hormonics

Show that the integral

$$\int d\Omega \; Y_{l_1}^{m_1}(\theta,\phi) \; Y_{l_2}^{m_2}(\theta,\phi) \; Y_{l_3}^{m_3}(\theta,\phi)$$

is proportional to the Clebsch-Gordan coefficient

$$(-1)^{m_3}\langle l_1 l_2 m_1 m_2 | l_1 l_2 l_3 - m_3\rangle \;,$$

where the proportionality constant is independent of $m_1$, $m_2$ and $m_3$.

Show that the integral vanishes if $l_1 + l_2 + l_3$ is an odd integer.

**Problem # 18.3 :** Zeeman and Paschenback effects

Consider the $^2$P multiplet of a hydrogen atom subject to spin-orbit coupling and an applied magnetic field along the $z$-direction

$$H = \mu_B B(l_z + 2s_z) + \xi \vec{l} \cdot \vec{s} \;.$$

(a) Obtain the energy eigenvalues $E$. Plot $E/\xi$ as a function of $\mu_B B/\xi$. Find the eigenfunctions in the limit $\mu_B B/\xi \to 0$ (Zeeman effect) and in the limit $\mu_B B/\xi \to \infty$ (Paschenback effect).

(b) $\xi = 0.225 cm^{-1}$ for the $^2$P-states in the 2p-shell of Li. What is the ratio $\mu_B B/\xi$ for 1 tesla?

**Problem # 18.4 :** Matrix elements and the Wigner-Eckart theorem

Let $\vec{J}_1$ and $\vec{J}_2$ be two angular momentum operators, $\vec{J} = \vec{J}_1 + \vec{J}_2$ is the sum of these two vectors, and $|JM\rangle$ denotes the eigenstates of $\vec{J}^2$ and $J_z$.

(a) Show that the matrix elements of $J_1^-$, $\langle JM|J_1^-|J'(M+1)\rangle$, vanish, unless $J' = J$ or $J' = J \pm 1$.

(b) Show also that the following expressions are independent of $M$

$$\frac{\langle JM|J_1^-|J(M+1)\rangle}{\sqrt{(J+M+1)(J-M)}} \; , \quad \frac{\langle JM|J_1^-|(J-1)(M+1)\rangle}{\sqrt{(J-M+1)(J-M)}} \; , \quad \frac{\langle JM|J_1^-|(J+1)(M+1)\rangle}{\sqrt{(J+M+2)(J+M+1)}} \; .$$

*Hints:* The Wigner-Eckart theorem is a useful tool to prove parts (a) and (b). Useful Clebsch-Gordan coefficients are

$$\langle J1(M+1)-1|J1JM\rangle = \sqrt{(J+M+1)(J-M)} \; ,$$

$$\langle (J-1)1(M+1)-1|(J-1)1JM\rangle = \sqrt{(J-M-1)(J-M)} \; ,$$

$$\langle (J+1)1(M+1)-1|(J+1)1JM\rangle = \sqrt{(J+M+1)(J+M+1)} \; .$$