QUANTUM MECHANICS

Homework set #20:

Relativistic Quantum Mechanics

Problem # 20.1 : One-dimensional Dirac equation with square-well potential

Consider the Hamiltonian for a Dirac particle of mass m moving in one spatial dimension under the influence of a scalar "square-well" potential,

$$H = -i\alpha \frac{d}{dx} + \beta[m + S(x)] \ ,$$

where $\alpha = \sigma_y$, $\beta = \sigma_z$, and the attractive scalar potential is given by

$$S(x) = \begin{cases} -S_0 & \text{for } |x| < L/2 \ , \\ 0 & \text{for } |x| > L/2 \ . \end{cases}$$

Use units in which $\hbar = c = 1$ throughout this problem.

(a) Show that if $\psi(x)$ is a solution of the Dirac equation ($H\psi = E\psi$) then so is the parity-transformed state, $\psi_{\mathrm{p}}(x) \equiv \sigma_z \psi(-x)$. Recall that Pauli matrices anticommute.

(b) Show that if $\psi(x)$ is a solution of the Dirac equation with energy $+E$, then the charge-conjugate state, $\psi_{\mathrm{c}}(x) \equiv \sigma_x \psi(x)$, is also a solution, but with energy $-E$.

(c) Propose a solution of the form,

$$\psi(x) = \begin{pmatrix} F(x) \\ G(x) \end{pmatrix} \ ,$$

and find the set of coupled first-order differential equations satisfied by F and G. Conclude that while F and G are continuous at $x = \pm L/2$, the first derivatives of both functions are not.

(d) Assuming that there is no degeneracy, state the conditions on F and G so that a solution with definite parity, i.e. $\psi_{\mathrm{p}}(x) = \pm\psi(x)$, might be found.

(e) Using your results from (c) and (d), find the conditions for which a positive-parity, positive-energy bound-state ($0 \le E < m$) solution exists. Note that you should state these conditions in the form of transcendental equations, but do not attempt to solve the equations.

Problem # 20.2 : Dirac equation for a massless particle

Consider the Dirac equation in Hamiltonian form for a massless particle, e.g. using units in which $\hbar = c = 1$,

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = (-i\vec{\alpha}\cdot\vec{\nabla})\psi(\vec{x},t) \ ,$$

where $\vec{\alpha} = \alpha_x\hat{x} + \alpha_y\hat{y} + \alpha_z\hat{z}$, and the α_i are matrices.

(a) Find the anticommutation relations that α_i must satisfy if each component of $\psi(\vec{x},t)$ is required to satisfy the Klein-Gordon equation,

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right)\psi(\vec{x},t) = 0 \ .$$

(b) Show that the 2×2 matrices $\alpha_i = \pm\sigma_i$ satisfy the conditions found in part (a). The σ_i are Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

(c) Look for satisfactory solutions to the Dirac equation when $\alpha_i = +\sigma_i$, of the form

$$\psi_+(\vec{x},t) = \begin{pmatrix} F_+ \\ G_+ \end{pmatrix} \exp(-iEt + i\vec{p}\cdot\hat{x}) \ ,$$

where $\vec{p}$ is the momentum of the particle and $E = |\vec{p}|$. Find the equations satisfied by $\begin{pmatrix} F_+ \\ G_+ \end{pmatrix}$.

(d) Look for stationary solutions to the Dirac equation when $\alpha_i = -\sigma_i$, of the form

$$\psi_-(\vec{x},t) = \begin{pmatrix} F_- \\ G_- \end{pmatrix} \exp(-iEt + i\vec{p}\cdot\hat{x}) \ ,$$

where $\vec{p}$ is the momentum of the particle and $E = |\vec{p}|$. Find the equations satisfied by $\begin{pmatrix} F_- \\ G_- \end{pmatrix}$.

(e) Explain which of the solutions of parts (c) and (d) would be appropriate to describe neutrinos, which are massless particles with negative helicity.

Problem # 20.3 : Zitterbewegung

Consider a free Dirac particle described by

$$H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2 \ ,$$

where $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The matrices $\vec{\sigma}$ and 1 are the 2×2 Pauli and unit matrices.

(a) Use the Heisenberg picture to establish

$$\frac{d}{dt}\vec{p} = 0 \quad , \quad \frac{d}{dt}\vec{\alpha} = \frac{2}{i\hbar}(c\vec{p} - H\vec{\alpha}) \quad , \quad \frac{d}{dt}\vec{r} = c\vec{\alpha} \ .$$

(b) These equations can be integrated to obtain

$$\vec{p} = \vec{p}(0) \qquad \vec{\alpha} = \frac{c\vec{p}}{H} + e^{2iHt/\hbar}\left[\vec{\alpha}(0) - \frac{c\vec{p}}{H}\right]$$
$$\vec{r} = \vec{r}(0) + \frac{c^2\vec{p}}{H}t + \frac{\hbar c}{2iH}\left\{e^{2iHt/\hbar} - 1\right\}\left[\vec{\alpha}(0) - \frac{c\vec{p}}{H}\right] \ .$$

Confirm this by differentiating the given results for $\vec{p}$, $\vec{\alpha}$, and $\vec{r}$.

(c) The last terms in the result for $\vec{\alpha}$ and $\vec{r}$ are related to the so-called "Zitterbewegung". These make no contribution to $\langle\vec{\alpha}\rangle$ and $\langle\vec{r}\rangle$ for wave packets containing solely positive or negative energy components. Prove this by establishing

$$\Lambda_{\pm}\left(\frac{\vec{p}}{H} - \vec{\alpha}\right)\Lambda_{\pm} = 0 \ ,$$

where $\Lambda_{\pm}$ are the projectors on states with positive and negative energy, respectively,

$$\Lambda_{\pm} = \frac{1}{2}\left[1 \pm \frac{H}{E_p}\right] \ ,$$

with $E_p = \sqrt{m^2c^4 + p^2c^2}$.

Hint: Start by establishing the general result $\Lambda[\Lambda, O]\Lambda = 0$ for any projection Λ and operator O, then consider $O = \vec{\alpha}$.

Problem # 20.4 : Spin-orbit coupling

The spin-orbit interaction in the hydrogen atom is given by the potential

$$V_{so} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}\vec{S}\cdot\vec{L} \ ,$$

which is of relativistic origin (i.e. it vanishes as $c \to \infty$). Here $\vec{S}$ and $\vec{L}$ are the spin and orbital angular momenta, respectively, and $V = -Ze^2/r$ is the Coulomb potential.

Using the wave-functions for the hydrogen atom as given by the solutions of the Schrödinger equation (non-relativistic hydrogen atom), obtain the spin-orbit coupling constant by calculating the matrix element

$$\langle nlm_lm_s|V_{so}|nlm_l'm_s'\rangle \ .$$

Obtain the energies of the hydrogen atom including the spin-orbit coupling and compare your result to the exact energies (solutions of the Dirac eqn.), given by

$$E_{nj} = -\frac{mc^2}{2}\frac{\alpha^2Z^2}{n^2}\left[1 + \frac{\alpha^2Z^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right) + \cdots\right] \ .$$

Here $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant.

Hint: You will need the expectation value

$$\langle nl|r^{-3}|nl\rangle = \frac{Z^3}{a_0^3n^3}\frac{1}{l(l+1/2)(l+1)} \ , \quad l \neq 0 \ ,$$

which can be calculated using the method of problem # 8.1 (calculation not required). Here $a_0 = \hbar^2/me^2$ is the Bohr radius for $Z = 1$.