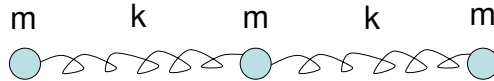


Theoretical Dynamics — PHY 5246

Final Exam December 14, 2006

1. (50 pts) Three objects of equal mass m are confined to move in one dimension. The masses are connected by springs with equal spring constants k and equilibrium lengths b , as shown in the figure below.

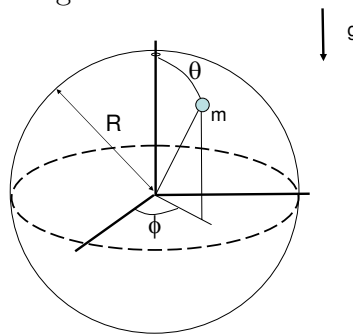


- (a) Determine the \mathbf{T} and \mathbf{V} matrices for which the Lagrangian for this system can be written in the form

$$L = \frac{1}{2} \dot{\eta}_i T_{ij} \dot{\eta}_j - \frac{1}{2} \eta_i V_{ij} \eta_j.$$

- (b) Obtain the frequencies of the normal modes for this system and determine the corresponding eigenvectors (don't bother to normalize them). Sketch the motion of the masses for each of these modes.
- (c) If the mass of the middle object is increased, which of the mode frequencies found in (b) will be changed?

2. (50 pts) Consider a particle of mass m confined to the surface of a sphere of radius R with gravity acting as shown in the figure below.



- (a) Obtain the Lagrangian for this system using the generalized coordinates θ and ϕ shown in the figure.
- (b) Find the momenta canonically conjugate to θ and ϕ and obtain the Hamiltonian for this system. Which of these quantities are conserved?
- (c) Obtain Hamilton's equations of motion for this system.

3. (50 pts) Recall that the Poisson bracket of two physical observables A and B is defined to be

$$[A, B] = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

(a) Prove that Hamilton's equations of motion imply that any physical observable $A(\{q_i\}, \{p_i\}, t)$ satisfies Poisson's equation,

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t},$$

where H is the Hamiltonian for the system.

(b) Consider a simple Harmonic oscillator with Hamiltonian

$$H = (p^2 + q^2)/2.$$

Use Poisson's equation to prove that the quantity $A = \tan^{-1}(q/p) - t$ is conserved for this system.

(c) Give a physical interpretation for this conserved quantity.

4. (50 pts) Consider the following transformation

$$q = \frac{\mathcal{Q}}{\mathcal{P}}; \quad p = \alpha \mathcal{P}^2.$$

(a) For what value of the parameter α is this transformation canonical?

(b) For the value of α obtain in (a), find a generating function of the second kind which generates this transformation.

(c) Apply this transformation to the Hamiltonian

$$H = p(q^2 + 1),$$

and obtain Hamilton's equations of motion for the transformed variables.

(d) Solve these equations to obtain general solutions for \mathcal{Q} and \mathcal{P} .

(e) Using the result of (d), obtain general solutions for the original variables q and p . Verify that these solutions satisfy Hamilton's equations of motion for the original Hamiltonian given in (c).