Theoretical Dynamics — PHY 5246

Final Exam December 9, 2004

1. Consider a particle with charge e moving in the plane in the presence of a perpendicular magnetic field B. If the vector potential is taken to be $\vec{A} = \frac{1}{2}\,\vec{r} \times \vec{B}$, then, in polar coordiantes, the Lagrangian for this particle is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}eBr^2\dot{\theta}.$$

(a) Find the momenta canonical to the coordinates r and θ for this system.

(b) Find the Hamiltonian for this system. You can do this either directly or using the matrix method.

(c) Find Hamilton's equations of motion for this system.

(d) What are the conserved quantities for this system?

2. Consider the following transformation

$$\mathcal{Q} = \frac{1}{2}(q^2 + p^2), \qquad \mathcal{P} = -\tan^{-1}\frac{q}{p}$$

(a) Show that this transformation is canonical using whatever method you prefer.

(b) Apply this transformation to the Hamiltonian

$$H = \frac{1}{2}(q^2 + p^2)$$

(c) Obtain Hamilton's equation of motion for the transformed variables and solve them to obtain the general solution for $\mathcal{Q}$ and $\mathcal{P}$ as a function of time.

(d) Obtain Hamilton's equations of motion for the untransformed variables q and p and solve them to obtain the general solution. Show that the result is consistent with your results for Part (c).

3. Consider a system described by the Hamiltonian

$$H = \frac{p^2}{2m} - kq.$$

(a) Write down the Hamilton-Jacobi equation for Hamilton's principal function $S(q, \alpha, t)$ for this system.

(b) Write down the Hamilton-Jacobi equation for Hamilton's characteristic function $W(q, \alpha)$, where $S(q, \alpha, t) = W(q, \alpha) - \alpha t$. What is the physical meaning of the separation constant α?

(c) Solve the Hamilton-Jacobi equation and obtain an expression for S. Use the result to obtain a general solution for q and p as a function of time.

(d) For the initial conditions $q(0) = q_0$ and $p(0) = 0$ find the subsequent motion of the system.

Hint:

$$\beta = \frac{\partial S}{\partial \alpha}, \qquad p = \frac{\partial S}{\partial q}$$