

PHY5246 Final Exam: Solution

Problem 1.

Part (a) The canonical momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad (1)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{eB}{2}r^2 \quad (2)$$

Part (b) The Lagrangian can be written

$$L = \frac{1}{2}(\dot{r}, \dot{\theta}) \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} + (0, \frac{1}{2}eBr^2) \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} \quad (3)$$

and so, using the matrix method, we find the Hamiltonian is

$$H = \frac{1}{2}(p_r, p_\theta - \frac{1}{2}eBr^2) \begin{pmatrix} (m)^{-1} & 0 \\ 0 & (mr^2)^{-1} \end{pmatrix} \begin{pmatrix} p_r \\ p_\theta - \frac{1}{2}eBr^2 \end{pmatrix} \quad (4)$$

or

$$H = \frac{1}{2m} \left(p_r^2 + \frac{1}{r^2} \left(p_\theta - \frac{eBr^2}{2} \right)^2 \right) \quad (5)$$

Part (c) Hamilton's equations of motion are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad (6)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{mr^2} \left(p_\theta - \frac{eBr^2}{2} \right) \quad (7)$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3} \left(p_\theta - \frac{eBr^2}{2} \right)^2 + \frac{eB}{mr} \left(p_\theta - \frac{eBr^2}{2} \right) \quad (8)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad (9)$$

Part (d) The conserved quantities are p_θ and H .

Problem 2.

Part (a) To verify that this transformation is canonical we need only evaluate the fundamental Poisson bracket,

$$[\mathcal{Q}, \mathcal{P}] = \frac{\partial \mathcal{Q}}{\partial q} \frac{\partial \mathcal{P}}{\partial p} - \frac{\partial \mathcal{Q}}{\partial p} \frac{\partial \mathcal{P}}{\partial q} \quad (10)$$

$$= (q) \left(\frac{q}{p^2} \frac{1}{1 + (q/p)^2} \right) - (p) \left(\frac{1}{p} \frac{-1}{1 + (q/p)^2} \right) \quad (11)$$

$$= \frac{q^2}{q^2 + p^2} + \frac{p^2}{q^2 + p^2} = 1 \quad (12)$$

Part (b) Applying this transformation to H yields

$$H = \mathcal{Q} \quad (13)$$

Part (c) Hamilton's equations of motion for the transformed problem are simply

$$\dot{Q} = \frac{\partial H}{\partial P} = 0 \quad (14)$$

$$\dot{P} = -\frac{\partial H}{\partial Q} = -1 \quad (15)$$

for which the general solution is

$$Q = Q_0 \quad (16)$$

$$P = -t + P_0 \quad (17)$$

where Q_0 and P_0 are constants.

Part (d) Hamilton's equations for the untransformed system are

$$\dot{q} = \frac{\partial H}{\partial p} = p \quad (18)$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -q \quad (19)$$

for which the general solution is

$$q = A \sin(t + \delta) \quad (20)$$

$$p = A \cos(t + \delta) \quad (21)$$

Thus

$$Q = \frac{1}{2}(q^2 + p^2) = A^2 \quad (22)$$

and

$$P = -\tan^{-1} \frac{q}{p} = -\tan^{-1}(\tan(t + \delta)) = -t - \delta \quad (23)$$

These results are consistent with Part (c), with $Q_0 = A^2$ and $P_0 = -\delta$.

Problem 3.

Part (a) The Hamilton-Jacobi equation for this system is

$$\frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 - kq + \frac{\partial S}{\partial t} = 0 \quad (24)$$

Part (b) Since H does not depend explicitly on time, we seek a solution of the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \quad (25)$$

which, when plugged into the Hamilton-Jacobi equation gives

$$\frac{1}{2m} \left(\frac{\partial W}{\partial q} \right)^2 - kq = \alpha \quad (26)$$

The separation constant α is equal to the Hamiltonian, and thus the total energy of the system.

Part (c) Solving for $\frac{\partial W}{\partial q}$ yields

$$\frac{\partial W}{\partial q} = \sqrt{(2m)(\alpha + kq)} \quad (27)$$

which we can integrate to obtain W (up to an irrelevant constant)

$$W = \int dq \sqrt{(2m)(\alpha + kq)} \quad (28)$$

Hamilton's principal function is then

$$S = \int dq \sqrt{(2m)(\alpha + kq)} - \alpha t \quad (29)$$

$$(30)$$

Since $\frac{\partial S}{\partial \alpha} = \beta$, where β is constant, we have

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{k} \sqrt{\alpha + kq} - t \quad (31)$$

Solving for q then yields

$$q = \frac{k}{2m}(\beta + t)^2 - \frac{\alpha}{k} \quad (32)$$

And, since $p = \frac{\partial S}{\partial q}$, we have

$$p = \frac{\partial S}{\partial q} = \sqrt{(2m)(\alpha + kq)} = k(\beta + t) \quad (33)$$

where the last equality is obtained by plugging in the above expression for q .

Part (d) From the initial conditions $q(0) = q_0$ and $p(0) = 0$ we have

$$p(0) = 0 = k\beta \Rightarrow \beta = 0 \quad (34)$$

and

$$q(0) = q_0 \Rightarrow q_0 = -\frac{\alpha}{k} \Rightarrow \alpha = -q_0 k \quad (35)$$

Thus the solution is

$$q(t) = q_0 + \frac{k}{2m} t^2 \quad (36)$$

$$p(t) = kt \quad (37)$$

Of course all we've done here is solved for the motion of a uniformly accelerating object and obtained the usual $\frac{1}{2}at^2$ expression.