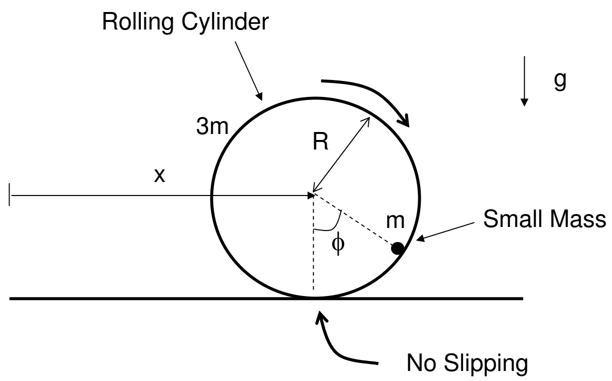
Theoretical Dynamics — PHY 5246

Final Exam December 15, 2005

1. (40 pts) Consider a uniform cylindrical shell of radius R and mass $3m$ rolling on a flat surface without slipping. A small object of mass m slides without friction inside the cylinder (see figure). Assume the small mass is confined to move in the plane of the page.



(a) Find the kinetic energy T and potential energy V for this system in terms of the generalized coordinates x and ϕ shown in the figure. Obtain the Lagrangian.

(b) Compute the canonical momenta p_x and p_ϕ conjugate to x and ϕ. Which (if any) of these are conserved? Is p_x equal to the x-component of the total linear momentum of the system?

(c) Obtain the Hamiltonian describing this system.

2. (30 pts) A system with one degree of freedom is described by the Hamiltonian

$$H = qp.$$

(a) Write down the Hamilton-Jacobi equation for this system.

(b) Using separation of variables, solve for Hamilton's principal function $S(q, \alpha, t)$.

(c) From $S(q, \alpha, t)$, obtain a general solution for q and p as a function of time. Verify that this solution satisfies Hamilton's equations of motion.

3. (30 pts) Consider the transformation

$$\mathcal{Q} = \frac{p}{q}, \qquad \mathcal{P} = \alpha q^2.$$

(a) For what value of the parameter α is this transformation canonical?

(b) For this value of α, find a generating function of the first kind which generates this transformation.

(c) Is it possible to find generating functions of all four kinds which generate this transformation? If not, indicate which kind(s) of generating functions cannot be found.

4. (40 pts) Consider a system described by the Lagrangian

$$L = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right),$$

where γ is a positive constant.

(a) Show that the Euler-Lagrange equation for this system implies

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0.$$

(Note: This is the equation of motion for a damped harmonic oscillator.)

(b) Find p, the momentum canonically conjugate to q, and show that the Hamiltonian for this system is

$$H = e^{-2\gamma t}\frac{p^2}{2m} + e^{2\gamma t}\frac{1}{2}m\omega^2 q^2.$$

Is this Hamiltonian conserved?

(c) Apply the canonical transformation generated by

$$F_2(q, \mathcal{P}, t) = e^{\gamma t} q\mathcal{P}$$

to the Hamiltonian obtained in (b). Is the transformed Hamiltonian conserved?

(d) Show that Hamilton's equations of motion for the transformed system are

$$\dot{\mathcal{Q}} = \frac{\mathcal{P}}{m} + \gamma\mathcal{Q}, \quad (1)$$

$$\dot{\mathcal{P}} = -m\omega^2\mathcal{Q} - \gamma\mathcal{P}. \quad (2)$$

(e) Combine these equations to find a single second order differential equation for $\mathcal{Q}$. (**Hint**: Take the time derivative of Eq. (1), then use Eqs. (1) and (2) to eliminate $\dot{\mathcal{Q}}$ and $\dot{\mathcal{P}}$.) For the case $\omega > \gamma$ find a general solution of this equation for $\mathcal{Q}(t)$.

(f) Using the result of (e), obtain a general solution for $q(t)$.

Hint: For a canonical transformation of the second kind with generating function $F_2(q, \mathcal{P}, t)$,

$$p = \frac{\partial F_2}{\partial q}, \quad \mathcal{Q} = \frac{\partial F_2}{\partial \mathcal{P}}, \quad K = H + \frac{\partial F_2}{\partial t}.$$