

PHY5246 Final Exam: Solution

1.

(a) The translational kinetic energy associated with the center-of-mass motion of the cylinder is

$$T_{trans.}^{cyl.} = \frac{1}{2}(3m)\dot{x}^2. \quad (1)$$

Due to the no slipping constraint, the angular velocity of the cylinder is $\omega = \dot{x}/R$. The rotational kinetic energy of the cylinder about its center of mass is then

$$T_{rot.}^{cyl.} = \frac{1}{2}(3m)R^2 \left(\frac{\dot{x}}{R}\right)^2 = \frac{1}{2}(3m)\dot{x}^2, \quad (2)$$

and the total kinetic energy of the cylinder is

$$T^{cyl.} = 3m\dot{x}^2. \quad (3)$$

The position of the small mass m in cartesian coordinates is

$$\vec{r} = (x + R \sin \phi)\hat{i} - R \cos \phi\hat{j}, \quad (4)$$

and so its velocity is

$$\dot{\vec{r}} = (\dot{x} + R\dot{\phi} \cos \phi)\hat{i} + R\dot{\phi} \sin \phi\hat{j}, \quad (5)$$

which has square magnitude

$$\dot{\vec{r}}^2 = (\dot{x} + R\dot{\phi} \cos \phi)^2 + R^2\dot{\phi}^2 \sin^2 \phi = \dot{x}^2 + R^2\dot{\phi}^2 + 2R\dot{x}\dot{\phi} \cos \phi. \quad (6)$$

Thus the kinetic energy of the small mass is

$$T^{mass} = \frac{1}{2}m(\dot{x}^2 + R^2\dot{\phi}^2 + 2R\dot{x}\dot{\phi} \cos \phi), \quad (7)$$

and the total kinetic energy is

$$T = T^{cyl.} + T^{mass} = \frac{1}{2}m(7\dot{x}^2 + R^2\dot{\phi}^2 + 2R\dot{x}\dot{\phi} \cos \phi). \quad (8)$$

Finally, since the potential energy of the small mass is

$$V = -mgR \cos \phi, \quad (9)$$

(the potential energy of the cylinder is, of course, constant, so we can ignore it), the Lagrangian for this system is

$$L = T - V = \frac{1}{2}m(7\dot{x}^2 + R^2\dot{\phi}^2 + 2R\dot{x}\dot{\phi} \cos \phi) + mgR \cos \phi. \quad (10)$$

(b) The momenta canonically conjugate to x and ϕ are readily found to be,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m(7\dot{x} + R\dot{\phi} \cos \phi), \quad (11)$$

and

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m(R^2\dot{\phi} + R\dot{x} \cos \phi). \quad (12)$$

 x is cyclic so p_x is conserved. ϕ is not cyclic so p_ϕ is not conserved.The x -component of the total linear momentum of this system is

$$p_x^{linear} = 3m\dot{x} + m(\dot{x} + R\dot{\phi} \cos \phi) = m(4\dot{x} + R\dot{\phi} \cos \phi), \quad (13)$$

which we can see is *not* equal to p_x . (Note that p_x is conserved, while p_x^{linear} is not, due to the frictional force required to maintain the no-slipping constraint.)

(c) Note that the Lagrangian for this system can be written

$$L = \frac{1}{2}(\dot{x}, \dot{\phi}) \mathbf{T} \begin{pmatrix} \dot{x} \\ \dot{\phi} \end{pmatrix} + L_0(x, \phi) \quad (14)$$

where

$$\mathbf{T} = m \begin{pmatrix} 7 & R \cos \phi \\ R \cos \phi & R^2 \end{pmatrix}, \quad (15)$$

and

$$L_0 = mgR \cos \phi. \quad (16)$$

(The \mathbf{a} vector which would be present if there were any linear dependence of L on the velocities \dot{x} and $\dot{\phi}$ is here equal to 0.)

To apply the matrix method we need only invert the matrix \mathbf{T} , with the result

$$\mathbf{T}^{-1} = \frac{1}{mR^2(7 - \cos^2 \phi)} \begin{pmatrix} R^2 & -R \cos \phi \\ -R \cos \phi & 7 \end{pmatrix}, \quad (17)$$

from which it follows that the Hamiltonian is,

$$H = \frac{1}{2}(p_x, p_\phi) \mathbf{T}^{-1} \begin{pmatrix} p_x \\ p_\phi \end{pmatrix} - L_0 = \frac{R^2 p_x^2 - 2R \cos \phi p_x p_\phi + 7 p_\phi^2}{2mR^2(7 - \cos^2 \phi)} - mgR \cos \phi. \quad (18)$$

2.

(a) The Hamilton-Jacobi equation for a system with one degree of freedom is

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0, \quad (19)$$

which, for $H = qp$ is

$$q \frac{\partial S}{\partial q} + \frac{\partial S}{\partial t} = 0. \quad (20)$$

(b) Since the Hamiltonian is time-independent we can use separation of variables and seek a generating function of the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t, \quad (21)$$

where $W(q, \alpha)$ is Hamilton's characteristic function. Plugging this into the Hamilton-Jacobi equation yields

$$q \frac{\partial W}{\partial q} = \alpha, \quad (22)$$

which can be readily solved to obtain

$$W = \int \frac{\alpha}{q} dq = \alpha \ln q. \quad (23)$$

(Note: here we have set the irrelevant integration constant to zero). Thus we have

$$S(q, \alpha, t) = \alpha(\ln q - t). \quad (24)$$

Here the separation constant α is, as usual, equal to H , which is conserved.

(c) Given $S(q, \alpha, t)$ we can find the general solution as follows,

$$\beta = \frac{\partial S}{\partial \alpha} = \ln q - t \Rightarrow q = e^{(t+\beta)} = Ae^t, \quad (25)$$

where $A = e^\beta$, and

$$p = \frac{\partial S}{\partial q} = \frac{\alpha}{q} \Rightarrow p = \frac{\alpha}{A} e^{-t}. \quad (26)$$

Hamilton's equations for the Hamiltonian $H = qp$ are

$$\dot{q} = \frac{\partial H}{\partial p} = q, \quad (27)$$

and

$$\dot{p} = -\frac{\partial H}{\partial q} = -p, \quad (28)$$

for which the general solution is $q = Ae^t$ and $p = Be^{-t}$, where $qp = AB = H = \alpha$, so $B = \alpha/A$. These are precisely the solutions obtained above by solving the Hamilton-Jacobi equation.

3.

(a) To find the value of the parameter α for which this transformation is canonical we must evaluate the fundamental Poisson bracket, for which we need the following partial derivatives,

$$\frac{\partial \mathcal{Q}}{\partial q} = -\frac{p}{q^2}; \quad \frac{\partial \mathcal{Q}}{\partial p} = \frac{1}{q}; \quad \frac{\partial \mathcal{P}}{\partial q} = 2\alpha q; \quad \frac{\partial \mathcal{P}}{\partial p} = 0. \quad (29)$$

Thus,

$$[\mathcal{Q}, \mathcal{P}]_{(q,p)} = \frac{\partial \mathcal{Q}}{\partial q} \frac{\partial \mathcal{P}}{\partial p} - \frac{\partial \mathcal{Q}}{\partial p} \frac{\partial \mathcal{P}}{\partial q} = \left(-\frac{p}{q^2}\right)(0) - \left(\frac{1}{q}\right)(2\alpha q) = -2\alpha, \quad (30)$$

and we see that for the transformation to be canonical we must have $\alpha = -1/2$.

(b) For $\alpha = -1/2$, to find a generating function of the first kind we first solve for p and \mathcal{P} in terms of q and \mathcal{Q} and set them equal to the appropriate partial derivatives of F_1 , with the result

$$p = q\mathcal{Q} = \frac{\partial F_1}{\partial q}, \quad (31)$$

and

$$\mathcal{P} = -\frac{q^2}{2} = -\frac{\partial F_1}{\partial \mathcal{Q}}. \quad (32)$$

Solving these equations for F_1 yields

$$F_1 = \frac{1}{2}q^2\mathcal{Q}. \quad (33)$$

(c) No. It is impossible to treat q and \mathcal{P} as independent variables because they are constrained by the equation $\mathcal{P} = \alpha q^2$. Therefore it is not possible to find a generating function of the second kind which generates this canonical transformation.

4.

(a) Turning the Lagrangian "crank" we have

$$\frac{\partial L}{\partial q} = -e^{2\gamma t} m\omega^2 q, \quad (34)$$

$$\frac{\partial L}{\partial \dot{q}} = e^{2\gamma t} m\dot{q}; \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 2\gamma e^{2\gamma t} m\dot{q} + e^{2\gamma t} m\ddot{q}, \quad (35)$$

and so the Euler-Lagrange equation is

$$e^{2\gamma t} (m\ddot{q} + 2\gamma m\dot{q} + m\omega^2 q) = 0, \quad (36)$$

or, after dividing through by $me^{2\gamma t}$,

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0. \quad (37)$$

(b) The momentum canonically conjugate to q is

$$p = \frac{\partial L}{\partial \dot{q}} = e^{2\gamma t} m \dot{q}, \quad (38)$$

and the Hamiltonian can be found easily using either the direct or matrix methods with the result

$$H = p\dot{q} - L = \frac{1}{2m} e^{-2\gamma t} p^2 + e^{2\gamma t} \frac{1}{2} m \omega^2 q^2. \quad (39)$$

Because H depends explicitly on time it is *not* conserved.

(c) Given the following generating function of the second kind

$$F_2(q, \mathcal{P}, t) = e^{\gamma t} q \mathcal{P}, \quad (40)$$

the corresponding canonical transformation is

$$p = \frac{\partial F_2}{\partial q} = e^{\gamma t} \mathcal{P} \quad (41)$$

$$\mathcal{Q} = \frac{\partial F_2}{\partial \mathcal{P}} = e^{\gamma t} q \rightarrow q = e^{-\gamma t} \mathcal{Q}, \quad (42)$$

and the transformed Hamiltonian is

$$K = H + \frac{\partial F_2}{\partial t} = \frac{\mathcal{P}^2}{2m} + \frac{1}{2} m \omega^2 \mathcal{Q}^2 + \gamma \mathcal{Q} \mathcal{P}. \quad (43)$$

This Hamiltonian has no explicit time dependence as so it *is* conserved.

(d) Hamilton's equations for the transformed system are then

$$\dot{\mathcal{Q}} = \frac{\partial K}{\partial \mathcal{P}} = \frac{\mathcal{P}}{m} + \gamma \mathcal{Q}, \quad (44)$$

and

$$\dot{\mathcal{P}} = -\frac{\partial K}{\partial \mathcal{Q}} = -m\omega^2 \mathcal{Q} - \gamma \mathcal{P}. \quad (45)$$

(e) To solve these equations we can take the derivative of both sides of (44) with respect to time t to obtain,

$$\ddot{\mathcal{Q}} = \frac{\dot{\mathcal{P}}}{m} + \gamma \dot{\mathcal{Q}}, \quad (46)$$

and then use (44) and (45) to eliminate $\dot{\mathcal{Q}}$ and $\dot{\mathcal{P}}$, with the result,

$$\ddot{\mathcal{Q}} = \frac{1}{m} (-m\omega^2 \mathcal{Q} - \gamma \mathcal{P}) + \gamma \left(\frac{\mathcal{P}}{m} + \gamma \mathcal{Q} \right) = -(\omega^2 - \gamma^2) \mathcal{Q} \quad (47)$$

For $\omega > \gamma$, the solution of this equation is

$$\mathcal{Q}(t) = A \sin(\sqrt{\omega^2 - \gamma^2} t + \delta). \quad (48)$$

(f) Using the fact that $q = e^{-\gamma t} \mathcal{Q}$, we have

$$q(t) = A e^{-\gamma t} \sin(\sqrt{\omega^2 - \gamma^2} t + \delta), \quad (49)$$

which is, of course, the correct solution for an underdamped harmonic oscillator.