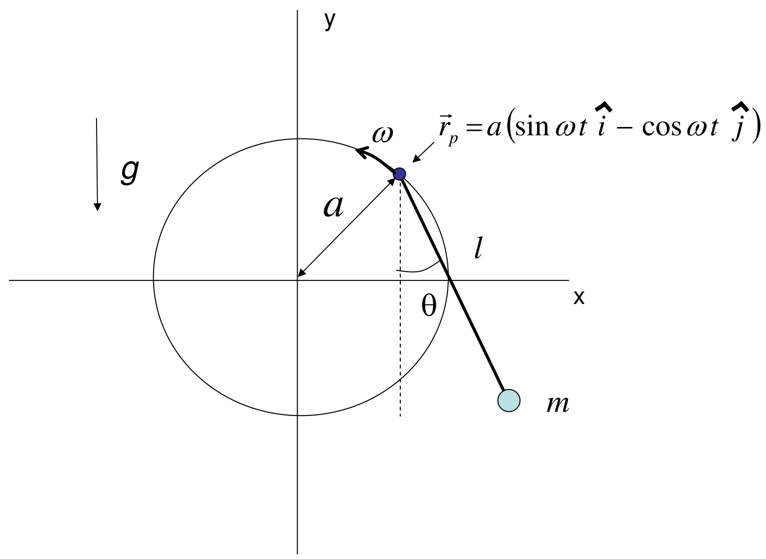
Theoretical Dynamics — PHY 5246

Midterm I October 20, 2006

1. (40 pts) (40 pts) A pendulum consists of a mass m attached to a massless rod of length l which hangs from a pivot point that is attached to a motor. The motor moves the pivot point $\vec{r}_p$ around a circle of radius a in the xy plane with constant angular velocity ω. The pendulum is confined to swing in the xy plane and gravity acts (see figure).



(a) Obtain the Lagrangian for this system in terms of the generalized coordinate θ shown in the figure.

(b) Find the momentum canonically conjugate to θ for this system. Is it conserved?

(c) Find the energy function h for this system. Is it equal to the mechanical energy? Is it conserved?

(d) Obtain the Euler-Lagrange equation of motion for this system.

2. (60 pts) A particle of mass m is attached to one end of a spring with equilibrium length a and spring constant k. The other end of the spring is fixed at the origin. If the displacement of the particle from the origin is $\vec{r}$, the potential energy for this system is then $V(r) = \frac{1}{2}k(r-a)^2$, and the Lagrangian is

$$L = \frac{1}{2}m\dot{\vec{r}}^{\,2} - \frac{1}{2}k(r-a)^2.$$

(a) Give a symmetry reason for why the angular momentum about the origin, $\vec{L}$, is conserved for this system. Show that if you choose the z-axis to be parallel to $\vec{L}$, so that $\vec{L} = l\hat{\mathbf{z}}$, then the motion of the particle takes place entirely in the xy plane.

(b) Given that the particle moves in the xy plane, express the Lagrangian for this system using the usual polar coordinates in the plane, r and θ, as generalized coordinates.

(c) Obtain the Euler-Lagrange equations for r and θ and combine them to obtain an effective one-dimensional problem for r.

(d) Determine the effective potential for this system and sketch it as a function of r. Describe the possible orbits (i.e. are they bounded or unbounded?).

(e) Now consider a circular orbit of radius $r_0 > a$. Find the angular frequency of this orbit. Express your answer in terms of r_0, k, a, and m.

(f) Find the angular frequency for small radial oscillations about this circular orbit. Again, express your answer in terms of r_0, k, a, and m.

(g) For what value of r_0/a is the angular frequency of small radial oscillations about the orbit equal to 4 times the angular frequency of the orbit? Sketch the shape of the orbit for this case.