PHY5246
Midterm 1: Solution

Problem 1.

Part (a)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \beta^2\dot{r}^2) - mg\beta r \tag{1}$$

Part (b)

$$\frac{\partial L}{\partial \dot{r}} = m(1+\beta^2)\dot{r} \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m(1+\beta^2)\ddot{r} \tag{3}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\beta \tag{4}$$

Euler-Lagrange equation for r

$$\Rightarrow m(1+\beta^2)\ddot{r} = mr\dot{\theta}^2 - mg\beta \tag{5}$$

θ is cyclic so

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l \tag{6}$$

is conserved.
Eliminating $\dot{\theta}$ from these equations gives the following effective 1D equation for the radial coordinate,

$$m(1+\beta^2)\ddot{r} = \frac{l^2}{mr^3} - mg\beta \tag{7}$$

Part (c)
Multiplying both sides by $\dot{r}$ and integrating on time

$$\int m(1+\beta^2)\ddot{r}\dot{r}dt = \int \left(\frac{l^2}{mr^3} - mg\beta\right)\dot{r}dt \tag{8}$$

we have

$$\int m(1+\beta^2)\dot{r}d\dot{r} = \int \left(\frac{l^2}{mr^3} - mg\beta\right)dr \tag{9}$$

which yields

$$\frac{1}{2}m(1+\beta^2)\dot{r}^2 = -\frac{l^2}{2mr^2} - mg\beta r + E \tag{10}$$

where the integration constant E is the total energy of the system.

Part (d)
For a circular orbit,

$$\dot{r} = 0 \Rightarrow mr_0\dot{\theta}^2 = m\frac{v_0^2}{r_0} = mg\beta \tag{11}$$

so the velocity is

$$\Rightarrow v_0 = \sqrt{r_0 g\beta} \tag{12}$$

Part (e)
Let $y = r - r_0$. Then, Taylor expanding, we have

$$m(1+\beta^2)\ddot{y} = -3\frac{l^2}{mr_0^4}y + O(y^2) = -\frac{3}{r_0}mg\beta y + O(y^2) \tag{13}$$

where we have used the fact that $l^2/(mr_0^3) = mg\beta$. For small oscillations we keep only linear terms and this equation describes a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{3g\beta}{r_0(1+\beta^2)}} \tag{14}$$

Part (f)
The period of the orbit is

$$\tau_{orb.} = \frac{2\pi r_0}{v_0} = 2\pi\sqrt{\frac{r_0}{g\beta}} \tag{15}$$

The period of the small oscillations is

$$\tau_{osc.} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0(1+\beta^2)}{3g\beta}} \tag{16}$$

Thus

$$\Rightarrow \frac{\tau_{orb.}}{\tau_{osc.}} = \sqrt{\frac{3}{1+\beta^2}} \tag{17}$$

Problem 2.
Part (a)

$$L = \frac{1}{2}(\dot{q}_1^2 + 2\dot{q}_2 q_1 + \dot{q}_2^2) - q_1^2 - \sin(t) \tag{18}$$

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1 \tag{19}$$

is not conserved, because q_1 is not cyclic.

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = q_1 + \dot{q}_2 \tag{20}$$

is conserved because q_2 is cyclic.
Part (b)
The energy function is

$$h = \dot{q}_1\frac{\partial L}{\partial \dot{q}_1} + \dot{q}_2\frac{\partial L}{\partial \dot{q}_2} - L \tag{21}$$

$$= \frac{1}{2}(\dot{q}_1 + \dot{q}_2) + q_1^2 + \sin(t) \tag{22}$$

It is not conserved (it depends explicitly on t) and not equal to the total energy (T is not a quadratic function of the $\dot{q}$'s, so $h \neq T + V$).