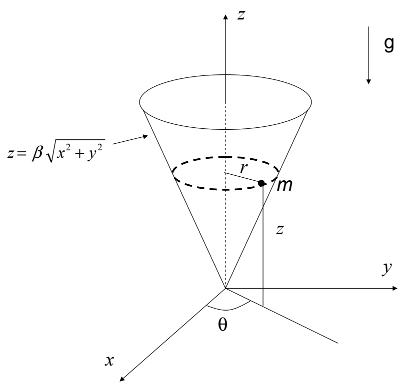
Theoretical Dynamics — PHY 5246

Midterm I   October 15, 2004

1. (60 pts) A small body of mass m moves under the influence of gravity on the interior of the surface of a cone. Friction can be ignored. Taking the z direction to be up, the surface of the cone is described by the equation

$$z = \beta\sqrt{x^2 + y^2}; \qquad \beta > 0.$$



(a) Write down the Lagrangian for this system in terms of the generalized coordinates r and θ, the polar coordinates in the xy plane, as shown in the figure.

(b) Find the Euler-Lagrange equations for r and θ, and combine them to obtain an effective one-dimensional problem for the radial coordinate r.

(c) The result of (b) should be a second-order differential equation for r. Obtain a first integral of this equation. What is the physical meaning of your result?

(d) Determine the condition on the initial velocity of the mass required to produce uniform circular motion of radius r_0. Express your answer in terms of β, g and r_0.

(e) Find the angular frequency of small radial oscillations about this circular motion. Express your answer in terms of β, g and r_0.

(f) Using the results of (d) and (e), compute the ratio of the period of the circular motion to the period of the small oscillations.

2. (20 pts) Consider a system described by two generalized coordinate q_1 and q_2 for which the kinetic energy (T) and potential energy (V) are given by

$$\begin{aligned} T(q_1, q_2, \dot{q}_1, \dot{q}_2, t) &= \frac{1}{2}(\dot{q}_1^2 + 2\dot{q}_2 q_1 + \dot{q}_2^2), \\ V(q_1, q_2, t) &= q_1^2 + \sin(t). \end{aligned}$$

(a) Form the Lagrangian for this system and find the generalized momenta conjugate to q_1 and q_2. State whether or not each is conserved.

(b) Construct the energy function h for this system. Is it conserved? Is it equal to the total energy of the system?