

## PHY5246 Midterm 1: Solution

Problem 1.

Part (a)

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \beta^2\dot{r}^2) - mg\beta r \quad (1)$$

Part (b)

$$\frac{\partial L}{\partial \dot{r}} = m(1 + \beta^2)\dot{r} \quad (2)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = m(1 + \beta^2)\ddot{r} \quad (3)$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\beta \quad (4)$$

Euler-Lagrange equation for  $r$

$$\Rightarrow m(1 + \beta^2)\ddot{r} = mr\dot{\theta}^2 - mg\beta \quad (5)$$

$\theta$  is cyclic so

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = l \quad (6)$$

is conserved.

Eliminating  $\dot{\theta}$  from these equations gives the following effective 1D equation for the radial coordinate,

$$m(1 + \beta^2)\ddot{r} = \frac{l^2}{mr^3} - mg\beta \quad (7)$$

Part (c)

Multiplying both sides by  $\dot{r}$  and integrating on time

$$\int m(1 + \beta^2)\ddot{r}\dot{r}dt = \int \left( \frac{l^2}{mr^3} - mg\beta \right) \dot{r}dt \quad (8)$$

we have

$$\int m(1 + \beta^2)\dot{r}d\dot{r} = \int \left( \frac{l^2}{mr^3} - mg\beta \right) dr \quad (9)$$

which yields

$$\frac{1}{2}m(1 + \beta^2)\dot{r}^2 = -\frac{l^2}{2mr^2} - mg\beta r + E \quad (10)$$

where the integration constant  $E$  is the total energy of the system.

Part (d)

For a circular orbit,

$$\dot{r} = 0 \Rightarrow mr_0\dot{\theta}^2 = m\frac{v_0^2}{r_0} = mg\beta \quad (11)$$

so the velocity is

$$\Rightarrow v_0 = \sqrt{r_0 g \beta} \quad (12)$$

Part (e)

Let  $y = r - r_0$ . Then, Taylor expanding, we have

$$m(1 + \beta^2)\ddot{y} = -3\frac{l^2}{mr_0^4}y + O(y^2) = -\frac{3}{r_0}mg\beta y + O(y^2) \quad (13)$$

where we have used the fact that  $l^2/(mr_0^3) = mg\beta$ . For small oscillations we keep only linear terms and this equation describes a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{3g\beta}{r_0(1 + \beta^2)}} \quad (14)$$

Part (f)

The period of the orbit is

$$\tau_{orb.} = \frac{2\pi r_0}{v_0} = 2\pi\sqrt{\frac{r_0}{g\beta}} \quad (15)$$

The period of the small oscillations is

$$\tau_{osc.} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0(1 + \beta^2)}{3g\beta}} \quad (16)$$

Thus

$$\Rightarrow \frac{\tau_{orb.}}{\tau_{osc.}} = \sqrt{\frac{3}{1 + \beta^2}} \quad (17)$$

Problem 2.

Part (a)

$$L = \frac{1}{2}(\dot{q}_1^2 + 2\dot{q}_2 q_1 + \dot{q}_2^2) - q_1^2 - \sin(t) \quad (18)$$

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1 \quad (19)$$

is not conserved, because  $q_1$  is not cyclic.

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = q_1 + \dot{q}_2 \quad (20)$$

is conserved because  $q_2$  is cyclic.

Part (b)

The energy function is

$$h = \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial L}{\partial \dot{q}_2} - L \quad (21)$$

$$= \frac{1}{2}(\dot{q}_1 + \dot{q}_2) + q_1^2 + \sin(t) \quad (22)$$

It is not conserved (it depends explicitly on  $t$ ) and not equal to the total energy ( $T$  is not a quadratic function of the  $\dot{q}$ 's, so  $h = T + V$ ).