PHY5246
Midterm 1: Solution

Problem 1.

Part (a)
Expressing the position vector of the mass in terms of the generalized coordinates we have

$$\vec{r} = (x\cos\alpha + l\sin\theta)\hat{i} - (x\sin\alpha + l\cos\theta)\hat{j} \tag{1}$$

The velocity is then

$$\dot{\vec{r}} = (\dot{x}\cos\alpha + \dot{\theta}l\cos\theta)\hat{i} - (\dot{x}\sin\alpha - \dot{\theta}l\sin\theta)\hat{j} \tag{2}$$

and the kinetic energy is

$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{\theta}^2 l^2 + 2\dot{x}\dot{\theta}l\cos(\alpha+\theta)) \tag{3}$$

The potential energy (up to an irrelevant constant) is

$$V = -mg(x\sin\alpha + l\cos\theta) \tag{4}$$

and so the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{\theta}^2 l^2 + 2\dot{x}\dot{\theta}l\cos(\alpha+\theta)) + mg(x\sin\alpha + l\cos\theta) \tag{5}$$

Part (b)

$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} + \dot{\theta}l\cos(\alpha+\theta)) \tag{6}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(\dot{\theta}l + \dot{x}l\cos(\alpha+\theta)) \tag{7}$$

p_x is conserved if $\alpha = 0$.

Part (c) Constraints have no explicit time dependence and there is no velocity dependent potential, so h is equal to the total energy. There is no explicit time dependence in L so h is conserved.

Problem 2.

Part (a) The kinetic energy of the two masses is

$$T = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 \tag{8}$$

and the potential energy is

$$V = m_2 g r \tag{9}$$

Thus the Lagrangian is

$$L = \frac{1}{2}m_1(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m_2\dot{r}^2 - m_2 g r \tag{10}$$

Part (b) Finding the Euler-Lagrange equations we compute

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = (m_1 + m_2)\ddot{r} \tag{11}$$

and

$$\frac{\partial L}{\partial r} = m_1 r\dot{\theta}^2 - m_2 g \tag{12}$$

Thus, the Euler-Lagrange equation for r is

$$\Rightarrow (m_1 + m_2)\ddot{r} = m_1 r\dot{\theta}^2 - m_2 g \tag{13}$$

θ is cyclic so

$$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = m_1 r^2\dot{\theta} = l \tag{14}$$

is conserved.
Eliminating $\dot{\theta}$ from these equations gives the following effective 1D equation for the radial coordinate,

$$(m_1 + m_2)\ddot{r} = \frac{l^2}{m_1 r^3} - m_2 g \tag{15}$$

Part (c)
Multiplying both sides of (15) by $\dot{r}$ and integrating over time yields

$$\int (m_1 + m_2)\ddot{r}\dot{r}dt = \int \left(\frac{l^2}{m_1 r^3} - m_2 g\right)\dot{r}dt \tag{16}$$

$$\int (m_1 + m_2)\dot{r}d\dot{r} = \int \left(\frac{l^2}{m_1 r^3} - m_2 g\right)dr \tag{17}$$

and, finally,

$$\frac{1}{2}(m_1 + m_2)\dot{r}^2 = -\frac{l^2}{2m_1 r^2} - m_2 g r + E \tag{18}$$

where the integration constant E is the total energy of the system.

Part (d)
For a circular orbit,

$$\dot{r} = 0 \Rightarrow m_1 r_0\dot{\theta}^2 = m_1\frac{v_0^2}{r_0} = m_2 g \tag{19}$$

where $v_0 = r_0\dot{\theta}$ is the speed of the particle, so

$$\Rightarrow v_0 = \sqrt{\frac{r_0 g m_2}{m_1}} \tag{20}$$

Part (e)
Let $y = r - r_0$. Then, Taylor expanding, we have

$$(m_1 + m_2)\ddot{y} = -3\frac{l^2}{m_1 r_0^4}y + O(y^2) = -\frac{3}{r_0}m_2 g y + O(y^2) \tag{21}$$

where we have used the fact that $l^2/(m_1 r_0^3) = m_2 g$. For small oscillations we keep only linear terms and this equation describes a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{3gm_2}{r_0(m_1 + m_2)}} \tag{22}$$

Part (f)

The period of small oscillations is

$$\tau_{osc} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0(m_1+m_2)}{3gm_2}} \tag{23}$$

and the period of the orbit is

$$\tau_{orbit} = \frac{2\pi r_0}{v_0} = 2\pi\sqrt{\frac{m_1 r_0}{m_2 g}} \tag{24}$$

The ratio is therefore

$$\frac{\tau_{osc}}{\tau} = \sqrt{\frac{m_1+m_2}{3m_2}} \tag{25}$$