# PHY5246
# Midterm 2: Solution

Problem 1.

Part (a)
From symmetry the $x$, $y$ and $z$ axes are the principal axes. The principal moments of inertia are then

$$\begin{aligned} I_{xx} &= 2ma^2 && (1)\\ I_{yy} &= 2m(2a)^2 = 8ma^2 && (2)\\ I_{zz} &= 2ma^2 + 2m(2a)^2 = 10ma^2 && (3) \end{aligned}$$

Part (b) The kinetic energy is

$$T = \frac{1}{2}(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2) = \frac{1}{2}\left(2ma^2\left(\frac{\omega}{\sqrt{2}}\right)^2 + 8ma^2\left(\frac{\omega}{\sqrt{2}}\right)^2\right) = \frac{1}{2}5ma^2\omega^2 \tag{4}$$

Since $T = \frac{1}{2}I(\hat{n})\omega^2$ we can simply read off the moment of inertia about the $(1,1,0)/\sqrt{2}$ axis passing through the center of mass

$$I_{CM} = 5ma^2 \tag{5}$$

Part (c) The parallel-axis theorem states that

$$I' = I_{CM} + MR_\perp^2 \tag{6}$$

where $M = 4m$ is the total mass of the body and $R_\perp = a/\sqrt{2}$ is the perpendicular distance between the two axes.

$$\Rightarrow I' = 5ma^2 + 4m\left(\frac{a}{\sqrt{2}}\right)^2 = 7ma^2 \tag{7}$$

Problem 2.

Part (a) To find the translational contribution to the kinetic energy of the system we need the position of the center of mass of the bar

$$\vec{r}_{cm} = (x + l\sin\theta)\hat{i} - l\cos\theta\hat{k} \tag{8}$$

from which we can find its velocity

$$\dot{\vec{r}}_{cm} = (\dot{x} + l\dot{\theta}\cos\theta)\hat{i} + l\dot{\theta}\sin\theta\hat{k} \tag{9}$$

and so

$$T_{trans} = \frac{1}{2}m\dot{\vec{r}}_{cm}^{\,2} = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + l^2\dot{\theta}^2) \tag{10}$$

The rotational kinetic energy is

$$T_{rot} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{3}ml^2\right)\dot{\theta}^2 \tag{11}$$

and the total kinetic energy is

$$T = T_{trans} + T_{rot} = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + \frac{4}{3}l^2\dot{\theta}^2) \tag{12}$$

The potential energy of the system is

$$V = \frac{1}{2}kx^2 - mgl\cos\theta \tag{13}$$

and so the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta + \frac{4}{3}l^2\dot{\theta}^2) - \frac{1}{2}kx^2 + mgl\cos\theta \tag{14}$$

Part (b) The kinetic energy of the system can be written

$$T = \frac{1}{2}(\dot{x},\dot{\theta})\begin{pmatrix} m & ml\cos\theta \\ ml\cos\theta & \frac{4}{3}ml^2 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} \tag{15}$$

and the potential energy can be written

$$V = -mgl + \frac{1}{2}kx^2 + \frac{1}{2}mgl\theta^2 + O(\theta^4) \tag{16}$$

$$= -mgl + \frac{1}{2}(x,\theta)\begin{pmatrix} k & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} + O(\theta^4) \tag{17}$$

The **T** matrix is found be setting $\theta = 0$ (the equilibrium value) in the matrix in (15)

$$\mathbf{T} = \begin{pmatrix} m & ml \\ ml & \frac{4}{3}ml^2 \end{pmatrix} \tag{18}$$

and the **V** matrix is

$$\mathbf{V} = \begin{pmatrix} k & 0 \\ 0 & mgl \end{pmatrix} \tag{19}$$

Part (c) To find the eigenfrequencies we solve the secular equation

$$\det(\mathbf{V} - \lambda\mathbf{T}) = \det\begin{pmatrix} k - \lambda m & -\lambda ml \\ -\lambda ml & mgl - \lambda\frac{4}{3}ml^2 \end{pmatrix} = 0 \tag{20}$$

$$(k - \lambda m)(mgl - \lambda\frac{4}{3}ml^2) - \lambda^2 m^2 l^2 = 0 \tag{21}$$

$$(k - \lambda m)(g - \lambda\frac{4}{3}l) - \lambda^2 ml = 0 \tag{22}$$

$$\frac{1}{3}ml\lambda^2 - \lambda(mg + \frac{4}{3}kl) + kg = 0 \tag{23}$$

$$\omega_\pm^2 = \lambda_\pm = \frac{mg + \frac{4}{3}kl \pm \sqrt{(mg + \frac{4}{3}kl)^2 - \frac{4}{3}mlkg}}{\frac{2}{3}ml} \tag{24}$$