PHY5246
Midterm 2: Solution

Problem 1.

Part (a) The rotational kinetic energy of the two bars about their centers of mass is

$$T_{Rot.} = \frac{1}{2}(\frac{1}{12}ML^2)(\dot{\theta}_1^2 + \dot{\theta}_2^2) \tag{1}$$

and the translational kinetic energy of the centers of mass is

$$T_{Trans.} = \frac{1}{2}M(\frac{L}{2})^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) \tag{2}$$

so the total kinetic energy of the system is

$$T = T_{Rot.} + T_{Trans.} = \frac{1}{2}(\frac{1}{3}ML^2)(\dot{\theta}_1^2 + \dot{\theta}_2^2) \tag{3}$$

The total potential energy of the system is the sum of the gravitational and spring potential energies

$$V = -Mg\frac{L}{2}(\cos\theta_1 + \cos\theta_2) + \frac{1}{2}k(a - \sqrt{L^2(\cos\theta_1 - \cos\theta_2)^2 + (a + L\sin\theta_1 - L\sin\theta_2)^2})^2 \tag{4}$$

Part (b) The **T** matrix is easily seen to be

$$\mathbf{T} = \frac{1}{3}ML^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

and expanding V to second order in θ_1 and θ_2 yields

$$V \simeq Const. + \frac{1}{2}\frac{MgL}{2}(\theta_1^2 + \theta_2^2) + \frac{1}{2}kL^2(\theta_1 - \theta_2)^2 \tag{6}$$

from which we find the **V** matrix is

$$\mathbf{V} = \begin{pmatrix} MgL/2 + kL^2 & -kL^2 \\ -kL^2 & MgL/2 + kL^2 \end{pmatrix} \tag{7}$$

Part (c) The secular equation is

$$\det \begin{pmatrix} MgL/2 + kL^2 - \lambda ML^2/3 & -kL^2 \\ -kL^2 & MgL/2 + kL^2 - \lambda ML^2/3 \end{pmatrix} = 0 \tag{8}$$

$$(MgL/2 + kL^2 - \lambda ML^2/3)^2 - (kL^2)^2 = 0 \tag{9}$$

for which the solutions are

$$\lambda = \frac{3g}{2L} + \frac{3k}{M} \pm \frac{3k}{M} \tag{10}$$

Next we find the corresponding (unnormalized) eigenvectors

$$\omega_1^2 = \lambda_1 = \frac{3g}{2L} \Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \mathbf{a}_1 = 0 \tag{11}$$

$$\Rightarrow \mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{12}$$

and

$$\omega_2^2 = \lambda_2 = \frac{3g}{2L} + \frac{6k}{M} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{a}_2 = 0 \tag{13}$$

$$\Rightarrow \mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{14}$$

Part (d) The general solution is then

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = Re\left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp -i\omega_1 t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \exp -i\omega_2 t \right) \tag{15}$$

Applying the initial conditions at $t = 0$ we have

$$\begin{pmatrix} \theta_1(0) \\ \theta_2(0) \end{pmatrix} = Re\left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

$$Re\ c_1 + Re\ c_2 = 0 \tag{17}$$
$$Re\ c_1 - Re\ c_2 = 0 \tag{18}$$

which implies

$$Re\ c_1 = Re\ c_2 = 0 \tag{19}$$

and

$$\begin{pmatrix} \dot\theta_1(0) \\ \dot\theta_2(0) \end{pmatrix} = Re\left(-i\omega_1 c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i\omega_2 c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \tag{20}$$

$$\omega_1 Im\ c_1 + \omega_2 Im\ c_2 = \Omega \tag{21}$$
$$\omega_1 Im\ c_1 - \omega_2 Im\ c_2 = 0 \tag{22}$$

which gives

$$c_1 = i\frac{\Omega}{2\omega_1} = i\frac{\Omega}{2\sqrt{3g/2L}} \tag{23}$$
$$c_2 = i\frac{\Omega}{2\omega_2} = i\frac{\Omega}{2\sqrt{3g/2L + 6k/m}} \tag{24}$$

Thus we have

$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \frac{\Omega}{2\omega_1} \sin\omega_1 t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{\Omega}{2\omega_2} \sin\omega_2 t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{25}$$

Problem 2.
Part (a) The principal moments of inertia of the disk are

$$I_{33} = \int_0^R r dr \int_0^{2\pi} d\theta \frac{M}{\pi R^2} r^2 = \frac{1}{2} M R^2 \tag{26}$$
$$I_{11} = I_{22} = \int_0^R r dr \int_0^{2\pi} d\theta \frac{M}{\pi R^2} (r\cos\theta)^2 = \frac{1}{4} M R^2 \tag{27}$$

Part (b)

$$\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \frac{1}{4} M R^2 \omega\, (0, \cos\alpha, 2\sin\alpha) \tag{28}$$

Part (c)

$$T = \frac{1}{2}(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2) = \frac{1}{8}MR^2\omega^2 \left(\sin^2\alpha + 2\cos^2\alpha\right) = \frac{1}{8}MR^2\omega^2 \left(1 + \cos^2\alpha\right) \tag{29}$$

Since $T = \frac{1}{2}I(\hat{n})\omega^2$ where $I(\hat{n})$ is the moment of inertia about an axis parallel to $\vec{\omega}$, we have

$$I(\hat{n}) = \frac{1}{4}MR^2(1 + \cos^2\alpha) \tag{30}$$

Part (d) Since for the disk $I_1 = I_2$ and $I_3 = 2I_1$ Euler's equations become

$$\dot{\omega}_3 = \frac{(I_1 - I_2)}{I_3}\omega_1\omega_2 = 0 \Rightarrow \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \omega\cos\alpha \tag{31}$$

and

$$\dot{\omega}_1 = \frac{(I_2 - I_3)}{I_1}\omega_3\omega_2 = -\omega\cos\alpha\ \omega_2 \tag{32}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\ \omega_1 = \omega\cos\alpha\omega_1 \tag{33}$$

These two equations can be combined to give

$$\ddot{\omega}_2 = -\omega^2\cos^2\alpha\ \omega_2 \tag{34}$$

for which the solution is (using the initial condition that $\omega_2 = \omega\sin\alpha$ at $t = 0$)

$$\omega_2 = \omega\sin\alpha\ \cos(t\ \omega\cos\alpha) \tag{35}$$

and

$$\omega_1 = \frac{\dot{\omega}_2}{\omega\cos\alpha} = -\omega\sin\alpha\ \sin(t\ \omega\cos\alpha) \tag{36}$$

$$\Rightarrow \vec{\omega}(t) = \omega\ (-\sin\alpha\ \sin(t\ \omega\cos\alpha), \sin\alpha\ \cos(t\ \omega\cos\alpha), \cos\alpha) \tag{37}$$