

## PHY5246 Midterm 2: Solution

Problem 1.

Part (a) The principal moments of inertia of the tile are

$$I_{33} = \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \frac{M}{L^2} (x^2 + y^2) = \frac{1}{6} ML^2 \quad (1)$$

$$I_{11} = I_{22} = \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \frac{M}{L^2} x^2 = \frac{1}{12} ML^2 \quad (2)$$

Part (b)

$$\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = \frac{1}{12} ML^2\omega (0, -\sin\theta, 2\cos\theta) \quad (3)$$

Part (c) Since the components of  $\vec{L}$  in the body system are constant,  $\left(\frac{d\vec{L}}{dt}\right)_{body} = 0$ . The torque is then

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_{space} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} \quad (4)$$

$$= \frac{1}{12} ML^2\omega (0, -\sin\theta, \cos\theta) \times (0, -\sin\theta, 2\cos\theta) \quad (5)$$

$$= -\frac{1}{12} ML^2 \sin\theta \cos\theta (1, 0, 0) \quad (6)$$

Problem 2.

Part (a) By the parallel-axis theorem, the moment of inertia of the hoop about the pivot point is

$$I_{hoop} = mR^2 + mR^2 = 2mR^2 \quad (7)$$

The kinetic energy of the hoop is therefore

$$T_{hoop} = mR^2\dot{\theta}^2 \quad (8)$$

The position of the bead, in terms of  $\theta$  and  $\phi$  is

$$\vec{r} = -R((\cos\theta + \cos\phi)\hat{i} + (\sin\theta + \sin\phi)\hat{j}) \quad (9)$$

and so the velocity of the bead is

$$\dot{\vec{r}} = -R(-(\dot{\theta}\sin\theta + \dot{\phi}\sin\phi)\hat{i} + (\dot{\theta}\cos\theta + \dot{\phi}\cos\phi)\hat{j}) \quad (10)$$

and its kinetic energy is

$$T_{bead} = \frac{1}{2} m \dot{\vec{r}}^2 \quad (11)$$

$$= \frac{1}{2} mR^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(\cos\theta\cos\phi + \sin\theta\sin\phi)) \quad (12)$$

$$= \frac{1}{2} mR^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)) \quad (13)$$

The total kinetic energy is therefore

$$T = T_{hoop} + T_{bead} = \frac{1}{2} mR^2 (3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)) \quad (14)$$

The potential energy of the hoop is  $V_{hoop} = -mgR \cos \theta$  and the potential energy of the bead is  $V_{bead} = -mgR(\cos \theta + \cos \phi)$  so the total potential energy is

$$V = V_{hoop} + V_{bead} = -mgR(2 \cos \theta + \cos \phi) \quad (15)$$

Part (b) We express the kinetic energy as

$$T = \frac{1}{2}mR^2(\dot{\theta}, \dot{\phi}) \begin{pmatrix} 3 & \cos(\theta - \phi) \\ \cos(\theta - \phi) & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} \quad (16)$$

Setting  $\theta$  and  $\phi$  to their equilibrium values ( $\theta = \phi = 0$ ) we see the  $\mathbf{T}$  matrix is

$$\mathbf{T} = mR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \quad (17)$$

and, expanding  $V$  to second order in  $\theta$  and  $\phi$  yields

$$V \simeq Const. + \frac{1}{2}mgR(2\theta^2 + \phi^2) \quad (18)$$

so the  $\mathbf{V}$  matrix is

$$\mathbf{V} = mgR \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad (19)$$

Part (c) The secular equation is  $\det(\mathbf{V} - \lambda\mathbf{T}) = 0$ . Dividing through by  $mgR$  and rescaling  $\lambda$  so that  $\omega^2 = \lambda \frac{g}{R}$  we have

$$\det \begin{pmatrix} 2 - 3\lambda & -\lambda \\ -\lambda & 1 - \lambda \end{pmatrix} = 0 \quad (20)$$

$$(2 - 3\lambda)(1 - \lambda) - \lambda^2 = 0 \quad (21)$$

$$2\lambda^2 - 5\lambda + 2 = 0 \quad (22)$$

$$\lambda = \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm 3}{4} \Rightarrow \lambda_1 = 2, \lambda_2 = \frac{1}{2} \quad (23)$$

Next we find the corresponding (unnormalized) eigenvectors

$$\omega_1^2 = \lambda_1 \frac{g}{R} = \frac{2g}{R} \Rightarrow \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \mathbf{a}_1 = 0 \quad (24)$$

$$\Rightarrow \mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad (25)$$

and

$$\omega_2^2 = \lambda_2 \frac{g}{R} = \frac{g}{2R} \Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{a}_2 = 0 \quad (26)$$

$$\Rightarrow \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (27)$$

Part (d) The general solution is

$$\begin{pmatrix} \theta(t) \\ \phi(t) \end{pmatrix} = Re \left( c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \exp -i\omega_1 t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp -i\omega_2 t \right) \quad (28)$$

Applying the initial conditions at  $t = 0$  we have

$$\begin{pmatrix} \theta(0) \\ \phi(0) \end{pmatrix} = Re \left( c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} \quad (29)$$

$$Re c_1 + Re c_2 = 0 \quad (30)$$

$$-2Re c_1 + Re c_2 = \phi_0 \quad (31)$$

which implies

$$Re c_1 = -\phi_0/3; \quad Re c_2 = \phi_0/3 \quad (32)$$

and

$$\begin{pmatrix} \dot{\theta}(0) \\ \dot{\phi}(0) \end{pmatrix} = Re \left( -i\omega_1 c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} - i\omega_2 c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (33)$$

$$\omega_1 Im c_1 + \omega_2 Im c_2 = 0 \quad (34)$$

$$-2\omega_1 Im c_1 + \omega_2 Im c_2 = 0 \quad (35)$$

which gives

$$Im c_1 = Im c_2 = 0 \quad (36)$$

and so

$$c_1 = -\frac{\phi_0}{3} \quad (37)$$

$$c_2 = \frac{\phi_0}{3} \quad (38)$$

Thus we have

$$\begin{pmatrix} \theta(t) \\ \phi(t) \end{pmatrix} = \frac{\phi_0}{3} \left( -\cos \omega_1 t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \cos \omega_2 t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \quad (39)$$