PHY5246
Midterm 2: Solution

1.

(a) The xy and yz planes passing through the point **O** are symmetry planes so the z and x axes are principal axes. The third principal axis must be orthogonal to both z and x, and thus is the y axis.

(b) The area of the triangle is

$$A = \int_0^h dy \int_{-y/2}^{y/2} dx = \frac{h^2}{2} \tag{1}$$

so its density is $M/(h^2/2)$. The moments of inertia about the x, y and z axes are then

$$\begin{aligned} I_x &= \frac{2M}{h^2}\int_0^h dy \int_{-y/2}^{y/2} dx\ y^2 = \frac{1}{2}Mh^2, && (2)\\ I_y &= \frac{2M}{h^2}\int_0^h dy \int_{-y/2}^{y/2} dx\ x^2 = \frac{2M}{h^2}\int_0^h dy 2\frac{(y/2)^3}{3} = \frac{1}{24}Mh^2, && (3)\\ I_z &= \frac{2M}{h^2}\int_0^h dy \int_{-y/2}^{y/2} dx\ (x^2+y^2) = I_x + I_y = \frac{13}{24}Mh^2. && (4) \end{aligned}$$

(c) The angular velocity vector $\vec{\omega}$ is parallel to the vector $(1,2,0)$. Thus, if the magnitude of the angular velocity is ω, the angular velocity vector is

$$\vec{\omega} = \omega(1,2,0)/\sqrt{5}. \tag{5}$$

The angular momentum is then

$$\vec{L} = (I_x\omega_x, I_y\omega_y, I_z\omega_z) = \frac{\sqrt{5}}{60}Mh^2(6,1,0), \tag{6}$$

and the rotational kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) && (7)\\ &= \frac{1}{2}\left(\left(\frac{1}{2}Mh^2\right)\frac{\omega^2}{5} + \left(\frac{1}{24}Mh^2\right)\frac{4\omega^2}{5}\right) && (8)\\ &= \frac{1}{20}Mh^2\left(1+\frac{1}{3}\right)\omega^2 && (9)\\ &= \frac{1}{15}Mh^2\omega^2. && (10) \end{aligned}$$

(d) By symmetry the x and z coordinates of the center of mass are 0. To find the y coordinate of the center of mass, y_{cm}, we must integrate over the body to obtain

$$y_{cm} = \frac{1}{h^2/2}\int_0^h dy \int_{-y/2}^{y/2} dx y = \frac{2}{h^2}\frac{h^3}{3} = \frac{2}{3}h \tag{11}$$

(e) By the same symmetry reasons given in Part (a), the principal axes about the center of mass are the x, y and z axes. However, the x and z axes passing through the center of mass are each displaced from the corresponding axes passing through the point **O** by a perpendicular distance of $2h/3$. The y axis passing through the point **O** also passes through the center of mass.

Applying the parallel axis theorem then yields the principal moments of inertia about the center of mass (denoted I'_α),

$$I'_x = I_x - M\left(\frac{2}{3}h\right)^2 = \frac{1}{18}Mh^2, \tag{12}$$

$$I'_y = I_y = \frac{1}{24}Mh^2, \tag{13}$$

$$I'_z = I_z - M\left(\frac{2}{3}h\right)^2 = \frac{7}{72}Mh^2. \tag{14}$$

(f) The principal axis corresponding to the intermediate moment of inertia is the x axis, so this is the unstable axis.

2.

(a) The center of mass of the rod is at

$$\vec{r}_{cm} = \frac{l}{2}(\cos\theta, \sin\theta), \tag{15}$$

so the translational kinetic energy of the rod is

$$T_{trans.} = \frac{1}{2}m\dot{\vec{r}}_{cm}^{\,2} = \frac{1}{2}m\left(\frac{l}{2}\right)^2\dot{\theta}^2. \tag{16}$$

The rotational kinetic energy of the rod about its center of mass is

$$T_{rot.} = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{\theta}^2, \tag{17}$$

and so the total kinetic energy of the rod is

$$T = T_{trans.} + T_{rot.} = \frac{1}{6}ml^2\dot{\theta}^2. \tag{18}$$

(b) The potential energy of the rod is

$$V = mg\frac{l}{2}\sin\theta, \tag{19}$$

and so the Lagrangian is

$$L = T - V = \frac{1}{6}ml^2\dot{\theta}^2 - mg\frac{l}{2}\sin\theta. \tag{20}$$

(c) The Euler-Lagrange equation for θ is found in the usual way.

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{3}ml^2\ddot{\theta}, \tag{21}$$

$$\frac{\partial L}{\partial\theta} = -mg\frac{l}{2}\cos\theta, \tag{22}$$

$$\Rightarrow \frac{1}{3}ml^2\ddot{\theta} = -mg\frac{l}{2}\cos\theta. \tag{23}$$

(d) The total energy (which is equal to h and is conserved for this problem) is

$$h = T + V = \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\sin\theta. \tag{24}$$

Given the initial condition that $\dot{\theta} = 0$ when $\theta = \theta_0$ we then have

$$\frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\sin\theta = \frac{1}{2}mgl\sin\theta_0. \tag{25}$$

When the rod hits the floor $\theta = 0$ and, solving for $\dot{\theta}$, we find

$$\dot{\theta} = -\sqrt{3\frac{g}{l}\sin\theta_0}. \tag{26}$$

(We choose the minus sign because θ is clearly decreasing just before it hits the floor.)