PHY5246

Theoretical Dynamics

Problem Set 12

Due: Friday, Dec 8, 2006

12.1 Consider a system for which the Hamiltonian has the form

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right).$$

(a) Obtain the Hamilton equations of motion for this system.

(b) Show that the transformation

$$\mathcal{Q} = pq^2, \qquad \mathcal{P} = \frac{1}{q},$$

is canonical and obtain a generating function.

(c) Apply this canonical transformation to H and obtain the Hamilton equations of motion for the transformed coordinate and momentum. Find the general solution of these equations.

(d) Using the result of (c) obtain expressions for the original coordinate and momentum q and p. Verify that these expressions satisfy the equations of motion you obtained in (a).

12.2 Consider a particle of mass m and charge q confined to move in the xy plane in the presence of a uniform magnetic field $\vec{B} = B\hat{z}$.

(a) Show that if we take the vector potential to be $\vec{A} = \frac{B}{2}\hat{z} \times \vec{r}$ the Hamiltonian for this particle is

$$H = \frac{1}{2m}\left[\left(p_x + \frac{qB}{2c}y\right)^2 + \left(p_y - \frac{qB}{2c}x\right)^2\right].$$

(b) By any method you choose, show that the following transformation is canonical for any value of the constant α:

$$x = \tfrac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right), \quad p_x = \tfrac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right),$$

$$y = \tfrac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right), \quad p_y = -\tfrac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right).$$

(c) Apply this transformation to the Hamiltonian given above, taking

$$\alpha^2 = \frac{qB}{c}.$$

(d) From the transformed Hamiltonian obtain the motion of the particle as a function of time.

12.3 Consider a system described by the Lagrangian

$$L = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right),$$

where γ is a positive constant.

(a) Show that the Euler-Lagrange equation for this system implies

$$\ddot{q} + 2\gamma\dot{q} + \omega^2 q = 0. \tag{1}$$

(Note: This is the equation of motion for a damped harmonic oscillator.)

(b) Find p, the momentum canonically conjugate to q, and obtain Hamiltonian for this system. Is it conserved?

(c) Apply the canonical transformation generated by

$$F_2(q, \mathcal{P}, t) = e^{\gamma t} q\mathcal{P}$$

to the Hamiltonian obtained in (b). Is the transformed Hamiltonian conserved?

(d) Obtain Hamilton's equations of motion for the transformed system. Combine these equations to find a single second order differential equation for $\mathcal{Q}(t)$, and obtain a general solution to this equation for the case $\omega > \gamma$. Using this solution, obtain the general solution to (1) for the original coordinate $q(t)$.

12.4 Consider a system for which the Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{mk}{q^2}.$$

(a) Write the Hamilton-Jacobi equation for this system and solve it to obtain the generating function $S(q, \mathcal{P}, t)$.

(b) Using your solution, find $q(t)$ and $p(t)$ subject to the initial conditions $p(0) = p_0 = 0$ and $q(0) = q_0$.