

**Physics 5492**  
**Condensed Matter Physics II**  
**Take-Home Final**  
Due: Friday, April 26, 2001

Instructions

- I The final is due in my mail box in the main office in the Keen building at noon on Thursday, April 26, 2001. There will be no extensions.
- II In completing the final you may consult any books you wish, and please feel free to ask me any clarifying questions.
- III *Please do not discuss the final with other members of the class until after the deadline.*
- IV Good luck!

1. **Critical Field for Surface Superconductivity:  $H_{c3}$ .**— In class we discussed the fact that in a type II superconductor a nonzero superconducting order parameter only appears for applied magnetic fields which are less than the upper critical field  $H_{c2}$ . In HW problem 4.2 you showed that

$$H_{c2} = \frac{\Phi_0}{2\pi\xi^2}$$

where  $\xi$  is the Ginzburg-Landau coherence length.

In this problem you will see that when the boundary of the superconducting material is taken into account there is a *third* critical field,  $H_{c3}$ , below which a nonzero order parameter first appears at the *surface* of the superconductor. (This result was first reported in D. Saint-James and P.G. deGennes, Phys. Letters **8**, 308 (1964).)

Recall the usual Ginzburg-Landau expression for the free energy density of the superconducting state relative to that of the normal state,

$$F_s - F_n = \alpha|\psi(\mathbf{r})|^2 + \frac{\beta}{2}|\psi(\mathbf{r})|^4 + \frac{1}{2m^*} \left| \left( \frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A} \right) \psi(\mathbf{r}) \right|^2.$$

The total free energy is then given by

$$\mathcal{F} = \int_{\Omega} (F_s - F_n) d^3r \quad (1)$$

where the integration is over the volume  $\Omega$  of the superconducting material.

- (a) Show that if  $\psi(\mathbf{r})$  is varied by  $\delta\psi(\mathbf{r})$  then the variation of  $\mathcal{F}$ , after performing the appropriate integration by parts, is given by

$$\delta\mathcal{F} = \int d^3r \left\{ \delta\psi^* \left[ \alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*} \left( \frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A} \right)^2 \psi \right] + \text{C.C.} \right\} \quad (2)$$

where C.C. denotes complex conjugate. Thus  $\mathcal{F}$  is extremized when  $\psi(\mathbf{r})$  satisfies the Ginzburg-Landau equation

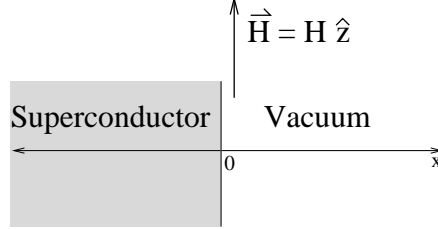
$$\frac{1}{2m^*} \left( \frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A} \right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0. \quad (3)$$

- (b) Show that the requirement that the boundary terms which appeared when you integrated by parts to derive (2) must vanish implies the following boundary condition at the surface of the superconductor

$$\hat{\mathbf{n}} \cdot \left( \frac{\hbar}{i}\nabla - \frac{e^*}{c}\mathbf{A} \right) \psi = 0 \quad (4)$$

where  $\hat{\mathbf{n}}$  is a unit vector normal to the surface.

Now consider the planar interface between a superconductor filling the half-infinite space to the left of the  $yz$  plane centered at the origin ( $x < 0$ ), and a vacuum to the right of this plane ( $x > 0$ ). Imagine that a magnetic field pointing in the  $\hat{z}$  direction,  $\mathbf{H} = H\hat{z}$ , is applied to the system. Work in the Landau gauge for which  $\mathbf{A} = (0, xH, 0)$ .



The problem is now to calculate the highest field for which it is possible to have a physically acceptable nonzero solution to (3) subject to the boundary condition (4) at the interface. This field is  $H_{c3}$ .

- (c) Using the same arguments you used for HW problem 4.2 justify the use of the linearized Ginzburg-Landau equations for this problem and show that the solutions of this equation which are uniform in the  $z$  direction ( $k_z = 0$ ) are of the form

$$\psi(\mathbf{r}) = Ae^{iky}\phi(x)$$

where

$$\left[ -\frac{\hbar^2}{2m^*} \frac{d^2}{dx^2} + \frac{1}{2m^*} \left( \hbar k - \frac{e^* H x}{c} \right)^2 \right] \phi(x) = |\alpha| \phi \quad (5)$$

with the boundary condition

$$\left. \frac{d\phi}{dx} \right|_{x=0} = 0. \quad (6)$$

- (d) A variational estimate of the ground state energy of the effective Schrödinger equation (5) can be found using a Gaussian wave function of the form

$$\phi(x) = \begin{cases} Ae^{-\rho x^2} & \text{for } x \leq 0, \\ 0 & \text{for } x > 0. \end{cases}$$

Show that  $\phi(x)$  automatically satisfies the boundary condition (6) and find the expectation value of the Hamiltonian in (5) using  $\phi(x)$ . Your result should depend on both the variational parameter  $\rho$  and the quantity  $k$  which appears explicitly in the Hamiltonian. Minimize the expectation value with respect to  $\rho$  and  $k$  to obtain the minimum value of  $|\alpha|$  for which a solution is possible.

- (e) Use the result of Part (d) to estimate the critical field  $H_{c3}$ . If all has gone well the result should be

$$H_{c3} = \left( 1 - \frac{2}{\pi} \right)^{-1/2} \frac{\Phi_0}{2\pi\xi^2} \simeq 1.7H_{c2}$$

where  $\xi = \hbar/(2m^*|\alpha|)^{1/2}$  is the coherence length and  $\Phi_0 = hc/e^* = hc/(2e)$  is the fluxoid.

**2. Anderson Impurity Model.**— When small amounts of a transition metal element are dissolved in a metallic host a localized magnetic moment may or may not form on the ion. (See Ashcroft and Mermin, pp. 685-686 for a brief discussion of this experimental fact.)

The following simple Hamiltonian, introduced by Anderson (P.W. Anderson, Phys. Rev. **124**, 41 (1961)), describes the essential physics of a single transition metal impurity dissolved in a metallic host in which, for simplicity, it is assumed that there is only one available atomic state associated with the impurity,

$$H = \sum_{\mathbf{k}\sigma} E_c(\mathbf{k}) c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + E_d \sum_{\sigma} d_{\sigma}^\dagger d_{\sigma} + \Delta \sum_{\mathbf{k}\sigma} \left( c_{\mathbf{k}\sigma}^\dagger d_{\sigma} + d_{\sigma}^\dagger c_{\mathbf{k}\sigma} \right) + U d_{\uparrow}^\dagger d_{\uparrow} d_{\downarrow}^\dagger d_{\downarrow}. \quad (7)$$

Here  $c_{\mathbf{k}\sigma}^\dagger$  creates a conduction electron with spin  $\sigma$  and momentum  $\mathbf{k}$  and  $d_{\sigma}^\dagger$  creates an electron on the impurity site with spin  $\sigma$ . The first term in (7) describes the band of conduction electrons, the second term describes the impurity level, the third term describes the tunneling of electrons from the impurity level to the conduction band and vice versa, and the fourth term describes the Coulomb energy cost of doubly occupying the impurity level.

An approximate solution to this model can be obtained using a similar technique to the Bogoliubov method for solving the superconductivity problem. Here this method is equivalent to the Hartree-Fock method.

(a) Let  $\langle n_{d\sigma} \rangle = \langle d_{\sigma}^\dagger d_{\sigma} \rangle$ . Using the identity

$$d_{\sigma}^\dagger d_{\sigma} = \langle n_{d\sigma} \rangle + \left( d_{\sigma}^\dagger d_{\sigma} - \langle n_{d\sigma} \rangle \right)$$

show that

$$H = H_{\uparrow} + H_{\downarrow} - U \langle n_{d\uparrow} \rangle \langle n_{d\downarrow} \rangle + O \left( (d_{\sigma}^\dagger d_{\sigma} - \langle n_{d\sigma} \rangle)^2 \right)$$

where

$$H_{\uparrow} = \sum_{\mathbf{k}} E_c(\mathbf{k}) c_{\mathbf{k}\uparrow}^\dagger c_{\mathbf{k}\uparrow} + E_d d_{\uparrow}^\dagger d_{\uparrow} + \Delta \sum_{\mathbf{k}} \left( c_{\mathbf{k}\uparrow}^\dagger d_{\uparrow} + d_{\uparrow}^\dagger c_{\mathbf{k}\uparrow} \right) + U \langle n_{d\downarrow} \rangle d_{\uparrow}^\dagger d_{\uparrow} \quad (8)$$

and

$$H_{\downarrow} = \sum_{\mathbf{k}} E_c(\mathbf{k}) c_{\mathbf{k}\downarrow}^\dagger c_{\mathbf{k}\downarrow} + E_d d_{\downarrow}^\dagger d_{\downarrow} + \Delta \sum_{\mathbf{k}} \left( c_{\mathbf{k}\downarrow}^\dagger d_{\downarrow} + d_{\downarrow}^\dagger c_{\mathbf{k}\downarrow} \right) + U \langle n_{d\uparrow} \rangle d_{\downarrow}^\dagger d_{\downarrow}. \quad (9)$$

In what follows we will assume the fluctuations of  $n_{d\sigma}$  about its mean value  $\langle n_{d\sigma} \rangle$  are small and ignore the  $O((d_{\sigma}^\dagger d_{\sigma} - \langle n_{d\sigma} \rangle)^2)$  term. The problem then decouples into two quadratic Hamiltonians for the up and down spin electrons,  $H_{\uparrow}$  and  $H_{\downarrow}$ , together with a self-consistency requirement.

(b) Show that the Hamiltonian (8) can be diagonalized so that it takes the following form

$$H_{\uparrow} = \sum_{\mathbf{k}} \left( E_+(\mathbf{k}) \gamma_{\mathbf{k}+}^\dagger \gamma_{\mathbf{k}+} + E_-(\mathbf{k}) \gamma_{\mathbf{k}-}^\dagger \gamma_{\mathbf{k}-} \right)$$

where

$$E_{\pm}(\mathbf{k}) = \frac{E_c(\mathbf{k}) + E_0 \pm \sqrt{(E_c(\mathbf{k}) - E_0)^2 + 4\Delta^2}}{2} \quad \text{and} \quad E_0 = E_d + U\langle n_{d\downarrow} \rangle,$$

and where the  $\gamma$  operators are given by

$$\gamma_{\mathbf{k}\mu}^\dagger = \alpha_{\mathbf{k}\mu} c_{\mathbf{k}\uparrow}^\dagger + \beta_{\mathbf{k}\mu} d_{\uparrow}^\dagger.$$

with

$$\beta_{\mathbf{k}\mu}^2 = \frac{\Delta^2}{(E_\mu(\mathbf{k}) - E_0)^2 + \Delta^2} \quad \text{and} \quad \alpha_{\mathbf{k}\mu}^2 = 1 - \beta_{\mathbf{k}\mu}^2.$$

Plot the energy bands  $E_+(\mathbf{k})$  and  $E_-(\mathbf{k})$  vs.  $\mathbf{k}$  and note that a small ‘hybridization’ gap has opened in the energy spectrum at  $E_0$ .

- (c) Show that the requirement that if the bands are completely filled so that every state is occupied then  $\langle n_{d\uparrow} \rangle = 1$  implies that

$$1 = \Omega \mathcal{P} \int_{-\infty}^{\infty} \frac{\Delta^2}{(E - E_0)^2} g(E) dE \simeq \Omega g(E_0) \mathcal{P} \int_{-\infty}^{\infty} \frac{\Delta^2}{(E - E_0)^2} dE \quad (10)$$

where  $\Omega$  is the volume of the system,  $g(E)$  is the density of states corresponding to the unperturbed band  $E_c(\mathbf{k})$ , and where the symbol  $\mathcal{P}$  indicates that the energy range around  $E_0$  corresponding to the gap in the spectrum you obtained in Part (b) is *not* included in the integration.

Now show that if the energy bands are not completely filled but are instead filled up to a given Fermi energy  $E_F$  then

$$\langle n_{d\uparrow} \rangle = \Omega \mathcal{P} \int_{-\infty}^{E_F} \frac{\Delta^2}{(E - E_0)^2} g(E) dE \simeq \Omega g(E_0) \mathcal{P} \int_{-\infty}^{E_F} \frac{\Delta^2}{(E - E_0)^2} dE. \quad (11)$$

Because the gap around  $E_0$  in the integrations in (10) and (11) is difficult to treat analytically, it is convenient to make the following approximation (which in no way changes the essential physics of the calculation). The approximation is to first introduce a parameter  $\epsilon$ , replacing the integrand in (10) and (11) with

$$\frac{\Delta^2}{(E - E_0)^2} \rightarrow \frac{\Delta^2}{(E - E_0)^2 + \epsilon^2}$$

and then to allow the integration to run over all energies. (The gap in the integration is no longer necessary because the parameter  $\epsilon$  cuts off the singularity in the integral at  $E_0$ .)

- (d) Using this approximation (10) becomes

$$1 = \Omega g(E_0) \int_{-\infty}^{\infty} \frac{\Delta^2}{(E - E_0)^2 + \epsilon^2} dE.$$

Evaluate this integral and obtain an expression for  $\epsilon$ .

Now evaluate the integral

$$\langle n_{d\uparrow} \rangle = \Omega g(E_0) \int_{-\infty}^{E_F} \frac{\Delta^2}{(E - E_0)^2 + \epsilon^2} dE$$

to obtain the following self-consistency condition relating  $\langle n_{d\uparrow} \rangle$  and  $\langle n_{d\downarrow} \rangle$ ,

$$\langle n_{d\uparrow} \rangle = \frac{1}{\pi} \operatorname{arccot} \frac{E_d - E_f + \langle n_{d\downarrow} \rangle U}{\epsilon} \quad (12)$$

(e) A similar analysis of (9) leads to a second self-consistency condition

$$\langle n_{d\downarrow} \rangle = \frac{1}{\pi} \operatorname{arccot} \frac{E_d - E_f + \langle n_{d\uparrow} \rangle U}{\epsilon}. \quad (13)$$

These equations can be solved graphically by plotting  $\langle n_{d\uparrow} \rangle$  vs.  $\langle n_{d\downarrow} \rangle$  for (12) and (13).

Study the graphical solutions of these equations and show that there are two possibilities: (1) There is only one solution for which  $\langle n_{d\uparrow} \rangle = \langle n_{d\downarrow} \rangle$  and no net magnetic moment forms on the ion; or (2) There are *three* solutions for two of which one finds  $\langle n_{d\uparrow} \rangle \neq \langle n_{d\downarrow} \rangle$  and a net magnetic moment forms on the ion.

**3. Plasma Analogy for Bilayer Quantum Hall Wave Functions.**— In addition to the single-layer quantum Hall systems we discussed in class, it is also possible to artificially engineer *bilayer* quantum Hall systems in which electrons are confined to two parallel two-dimensional electron gases with a typical separation of a few hundred angstroms. Because of the interlayer correlations in these systems new types of fractional quantum Hall states can (and do!) form in such bilayer systems.

Consider such a bilayer quantum Hall system in a perpendicular magnetic field  $B$  and assume the electrons are fully spin polarized by the field. The complex coordinates of electrons in layer 1 will be denoted  $z$  and those in layer 2 by  $w$ . Assume there is no tunneling between the layers so that electrons in different layers are effectively distinguishable. It is then possible to consider the following bilayer generalization of Laughlin's wave function,

$$\Psi_{mmn} = \left( \prod_{i<j} (z_i - z_j)^m \right) \left( \prod_{i<j} (w_i - w_j)^m \right) \left( \prod_{i,j} (z_i - w_j)^n \right) \exp \left( - \sum_{i=1}^{N_1} \frac{|z_i|^2}{4l_0^2} - \sum_{i=1}^{N_2} \frac{|w_i|^2}{4l_0^2} \right)$$

where  $l_0 = (\hbar c/eB)^{1/2}$  is the magnetic length. The requirement that the wave function be antisymmetric under the interchange of any two particles within a given layer implies that  $m$  must be odd.

- (a) Following Laughlin's plasma analogy write the square modulus of the wave function as

$$|\Psi_{mmn}|^2 = \exp -\beta E(z_1, \dots, z_{N_1}; w_1, \dots, w_{N_2}) \quad \text{where} \quad \beta = \frac{1}{m}$$

and find an expression for  $E(z_1, \dots, z_{N_1}; w_1, \dots, w_{N_2})$ .

- (b) Interpret the function  $E$  you obtained in Part (a) as the energy of a classical two-dimensional *two*-component plasma. (Hint: To do this you will have to assign *two* charges to each particle – one charge which only the  $z$  particles feel and another charge which only the  $w$  particles feel.)
- (c) Using the requirement of charge neutrality for both types of charges show that if  $m \neq n$  then

$$N_1 = N_2.$$

What happens when  $m = n$ ?

- (d) By convention, the total Landau level filling factor of a bilayer system is taken to be the sum of the filling factors in each layer. Again using the plasma analogy and the requirement of charge neutrality show that the total filling factor corresponding to  $\Psi_{mmn}$  is

$$\nu = \frac{2}{m+n}.$$