Physics 5670
Quantum Many-Body Physics
Take-Home Final
Due: Friday, May 2, 2003

Instructions

I The final is due in my mail box in the main office in the Keen building at 5:00 PM on Friday, May 2, 2003. There will be no extensions.

II In completing the final you may consult any books you wish, and please feel free to ask me any clarifying questions.

III *Please do not discuss the final with other members of the class until after the deadline.*

IV Good luck!

1. **Potential Scattering with Green's Functions.**
Consider a three-dimensional system of spinless non-interacting fermions in a random *one-body* potential

$$V(\mathbf{r}) = \frac{1}{\Omega}\sum_{\mathbf{q}} \tilde{V}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}}.$$

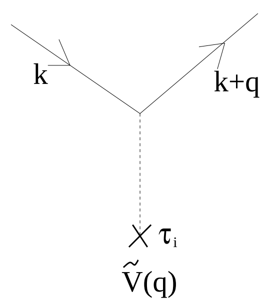
Here Ω is the volume of the system.

(a) Write down the second-quantized Hamiltonian for this system in the momentum representation.

(b) Obtain a formal expression for the partition function, Z, for this system in terms of a coherent-state functional integral.

(c) Formally expand the functional integral obtained in (b) in powers of V and obtain an expression for Z/Z_0 in terms of an infinite sum of correlation functions which can be evaluated using Wick's theorem. Here Z_0 is the unperturbed partition function. Your result should be analogous, but not identical, to Eq. 2.96 in Negele and Orland.

(d) Derive the Feynman rules for labeled diagrams corresponding to the Wick contractions which appear in the expansion obtained in (c). Note that since the perturbation V is a *one-body* potential, the diagram corresponding to the vertex will have the form,



Thus there are no Left-Right labels. However, there are τ labels.

(e) Discuss the symmetry factors for these diagrams and derive the Feynman rules for unlabeled diagrams in the expansion of Z/Z_0.

(f) Discuss the linked cluster theorem for this expansion and obtain expressions for the first and second-order contributions to the free energy in powers of V.

Now consider the exact 1-body Green's function for this system,

$$G(\mathbf{k}\tau_1|\mathbf{k}'\tau_2) = \langle T[a_{\mathbf{k}}^{(H)}(\tau_1) a_{\mathbf{k}}^{(H)\dagger}(\tau_2)]\rangle.$$

Note that due to scattering by the potential $\tilde{V}(\mathbf{q})$ the exact Green's function will not be diagonal in $\mathbf{k}$.

(g) Write a formal expression for $G(\mathbf{k}\tau_1|\mathbf{k}'\tau_2)$ as a ratio of coherent state path integrals.

(h) Expand the result of (g) in powers of V and obtain the Feynman rules for unlabeled diagrams contributing to G. Discuss symmetry factors and the cancellation of unlinked diagrams.

(i) Determine the Feynman rules for the Matsubara Fourier transform of G,

$$G(\mathbf{k},\mathbf{k}';\omega_n) = \int_0^\beta d\tau e^{i\omega_n\tau} G(\mathbf{k}\tau|\mathbf{k}'0),$$

in the frequency and momentum representation.

(j) Define the proper self-energy $\Sigma(\mathbf{k},\mathbf{k}';\omega_n)$ for this system. Note that because $\mathbf{k}$ is not conserved, $\Sigma(\mathbf{k},\mathbf{k}';\omega_n)$, like G, is not diagonal in $\mathbf{k}$. You should find that there is only one diagram contributing to $\Sigma(\mathbf{k},\mathbf{k}';\omega_n)$, thus you should be able to evaluate Σ exactly.

(k) Obtain Dyson's equation for this system. This should take the form of an integral equation for $G(\mathbf{k},\mathbf{k}';\omega_n)$.

2. **Exchange Contribution to the Random-Phase Approximation**.
In class we showed that the density response function

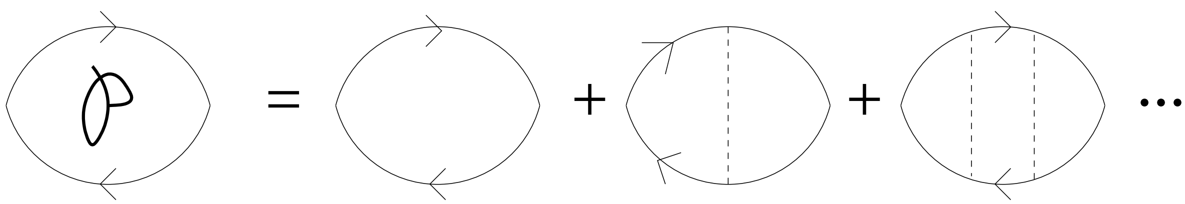
$$D(\mathbf{q}, \nu_n) = \int_0^\beta d\tau e^{i\nu_n t} \langle T[\hat{n}_H(-\mathbf{q}, t)\hat{n}_H(\mathbf{q}, t)] \rangle$$

can be expressed in terms of the proper polarization function $\mathcal{P}(\mathbf{q}, \nu_n)$ as

$$D(\mathbf{q}, \nu_n) = \frac{\mathcal{P}(\mathbf{q}, \nu_n)}{1 + \tilde{V}(\mathbf{q})\mathcal{P}(\mathbf{q}, \nu_n)}.$$

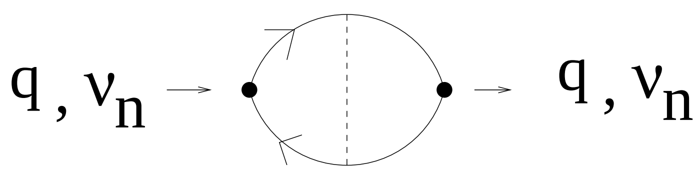
Here $\nu_n = 2\pi n/\beta$ is a bosonic Matsubara frequency.

A common approximation for $\mathcal{P}(q, \nu_n)$ is to sum the following set of diagrams



The random-phase approximation worked out in class consists of just keeping the first term in this series.

(a) Consider the case of spin-S fermions interacting via a spin-independent two-body interaction $\tilde{V}(\mathbf{q})$. Assign internal labels and write down an expression for the second term in this series, i.e. the diagram



and perform all the frequency sums.

(b) Now consider the special case of a delta function interaction $V(\mathbf{r}_1 - \mathbf{r}_2) = V_0\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$ for which $\tilde{V}(\mathbf{q}) = V_0$ is independent of q. Show that for this case it is possible to sum all the diagrams shown above, and obtain the resulting expression for $\mathcal{P}(\mathbf{q}, i\nu_n)$ and $D(\mathbf{q}, i\nu_n)$. (Hint: You should find that the nth order diagram contributing to this sum involves a factor of $(\mathcal{P}_0(\mathbf{q}, \nu_n))^{n+1}$ where $\mathcal{P}_0$ is the zeroth order polarization bubble evaluated in class. The sum is therefore similar to the standard RPA sum we did in class. The main difference is due to the lack of the fermion loop minus sign.)

(c) What happens for the case of spinless fermions ($S = 0$)?

3. **Magnetic Response of a Spin-1/2 Fermi Gas**.
Let $\hat{\sigma}_z(\mathbf{r})$ be the spin density operator for a gas of spin-1/2 fermions, defined to be

$$\hat{\sigma}_z(\mathbf{r}) = \hat{n}_\uparrow(\mathbf{r}) - \hat{n}_\downarrow(\mathbf{r}),$$

where $\hat{n}_\uparrow(\mathbf{r})$ and $\hat{n}_\downarrow(\mathbf{r})$ are the particle density operators for up and down spin particles, respectively. In second quantized notation

$$\hat{\sigma}_z(\mathbf{q}) = \sum_{\mathbf{k}} (a^\dagger_{\mathbf{k}+\mathbf{q},\uparrow} a_{\mathbf{k},\uparrow} - a^\dagger_{\mathbf{k}+\mathbf{q},\downarrow} a_{\mathbf{k},\downarrow}).$$

Now consider applying a time and space dependent magnetic field,

$$\mathcal{H}(\mathbf{r},t) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \mathcal{H}(\mathbf{q},t),$$

to the system. The corresponding perturbation,

$$\hat{H}'(t) = -\mu_0 \frac{1}{\Omega} \sum_{\mathbf{q}} \hat{\sigma}_z(\mathbf{q}) \mathcal{H}(\mathbf{q},t),$$

will result in an induced magnetization given by

$$M_z(\mathbf{q},t) = \mu_0 \langle \hat{\sigma}_z(\mathbf{q},t) \rangle.$$

From our general discussion of the fluctuation-dissipation theorem we know that

$$M_z(\mathbf{q},t) = -\mu_0^2 \int_{-\infty}^{\infty} D_R^\sigma(q,t-t') \mathcal{H}(\mathbf{q},t'),$$

where

$$D_R^\sigma(\mathbf{q},t-t') = -i\frac{1}{\Omega}\Theta(t-t')\langle \hat{\sigma}_z^{(H)}(\mathbf{q},t), \hat{\sigma}_z^{(H)}(-\mathbf{q},t')]\rangle,$$

or, if we perform the frequency Fourier transform,

$$M_z(\mathbf{q},\omega) = -\mu_0^2 D_R^\sigma(\mathbf{q},\omega)\mathcal{H}(\mathbf{q},\omega),$$

where

$$M_z(\mathbf{q},\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} M_z(\mathbf{q},t),$$

and

$$D_R^\sigma(q,\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} D_R^\sigma(q,t).$$

To calculate D_R^σ we must first calculate the corresponding imaginary-time Green's function

$$D^\sigma(q,\tau-\tau') = \frac{1}{\Omega}\langle T[\hat{\sigma}_z^{(H)}(\mathbf{q},\tau)\hat{\sigma}_z^{(H)}(-\mathbf{q},\tau')]\rangle,$$

with Matsubara transform

$$D^\sigma(q,\nu_n) = \int_0^\beta d\tau e^{i\nu_n\tau} D^\sigma(q,\tau),$$

and then analytically continue the result to obtain $D_R^\sigma(q,\omega)$.

(a) Derive the spectral representations of $D_R^\sigma(q,\omega)$ and $D^\sigma(q,\nu_n)$ and show that they are determined by the same spectral weight function.

(b) Use Wick's theorem to evaluate $D^\sigma(q,\nu_n)$ for a noninteracting system.

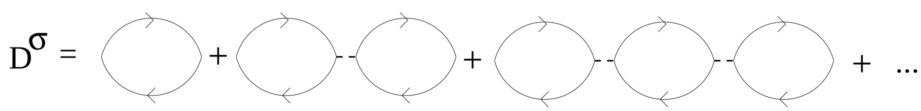
(c) Using the result of (a), analytically continue the expression for $D^\sigma(q,\nu_n)$ obtained in (b) to determine the real frequency response $D_R^\sigma(q,\omega)$.

(d) Find the generalized susceptibility of a noninteracting system in a static magnetic field $\chi(\mathbf{q},\omega=0) = M_z(\mathbf{q},\omega=0)/\mathcal{H}(\mathbf{q},\omega=\mathbf{0})$ and verify that

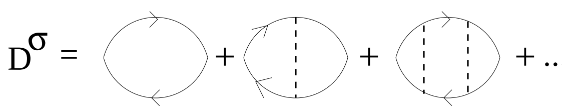
$$\lim_{q\to 0}\chi(q,0) = \begin{cases} 3\mu_0^2 n/(2E_F), & k_BT \ll E_F \quad \text{(Pauli paramagnetism)} \\ \mu_0^2 n/(k_BT), & k_BT \gg E_F \quad \text{(Curie paramagnetism)} \end{cases}$$

Here E_F is the Fermi energy at $T=0$ and n is the particle density.

(e) Now consider turning on a spin-independent two-body interaction $\tilde{V}(\mathbf{q})$. Discuss the diagrammatic expansion for D^σ. Note: It is similar to that for the density-response function discussed in class — the only difference is due to factors of -1 which appear in the spin sums. Show that because of this difference only the first term of the RPA series shown below is nonzero.



(f) Now consider the following series of diagrams (analogous to those studied in Problem 2).



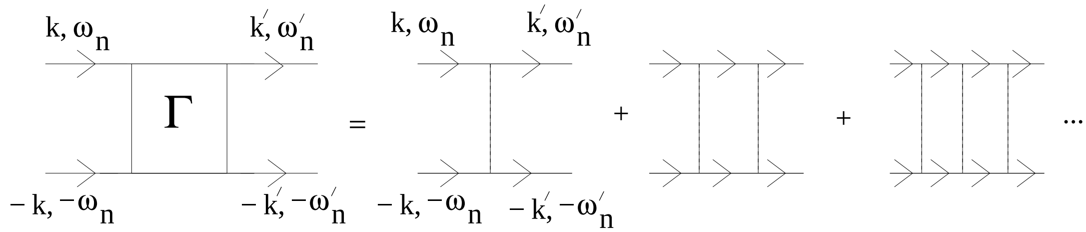
Show that these diagrams do *not* vanish, and hence do contribute to D^σ.

(g) For the case of a delta-function repulsive interaction, $\tilde{V}(\mathbf{q}) = V_0 > 0$, sum the diagrams shown in Part (f). (Hint: This can be done in the same way as in Problem 2). Analytically continue your result to obtain D_R^σ. Show that the static uniform magnetic susceptibility $\chi = \lim_{q\to 0}\chi(\mathbf{q},0)$ diverges when

$$V_0\frac{3n}{2E_F} = 1.$$

This divergence signals a ferromagnetic instability (Stoner criterion).

4. **Ladder Diagrams and Superconductivity**.
In the presence of an attractive interaction, a gas of fermions can undergo a finite temperature phase transition into a superconducting state. This phase transition is signaled by the divergence of the following sum of "ladder diagrams",



The scattering function $\mathbf{\Gamma}(\mathbf{k}, \mathbf{k}')$ is defined to be the sum of these diagrams, not including the four bare Green's function factors for the external legs.

(a) Show that the first three contributions to Γ are

$$\begin{aligned}\Gamma(\mathbf{k}, \mathbf{k}') &= -\tilde{V}(\mathbf{k}-\mathbf{k}') + \int \frac{d^3k_1}{(2\pi)^3}\tilde{V}(\mathbf{k}-\mathbf{k}_1)\tilde{V}(\mathbf{k}_1-\mathbf{k}')\frac{1}{\beta}\sum_{n_1} G_0(k_1, \omega_{n_1})G_0(-k_1, -\omega_{n_1}) \\ &\quad - \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}\tilde{V}(\mathbf{k}-\mathbf{k}_1)\tilde{V}(\mathbf{k}_1-\mathbf{k}_2)\tilde{V}(\mathbf{k}_2-\mathbf{k}') \\ &\qquad \times \frac{1}{\beta}\sum_{n_1} G_0(\mathbf{k}_1, \omega_{n_1})G_0(-\mathbf{k}_1, -\omega_{n_1})\frac{1}{\beta}\sum_{n_2} G_0(\mathbf{k}_2, \omega_{n_2})G_0(-\mathbf{k}_2, -\omega_{n_2}) \\ &\qquad + \cdots\end{aligned}$$

(b) Use the Matsubara trick to evaluate the frequency sum

$$\frac{1}{\beta}\sum_n G_0(\mathbf{k}, \omega_n)G_0(-\mathbf{k}, -\omega_n),$$

which appears in Γ.

(c) Consider the case of an attractive delta function interaction $V(\mathbf{r}_1 - \mathbf{r}_2) = -V_0\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$ so that $\tilde{V}(\mathbf{q}) = -V_0$ is independent of $\mathbf{q}$. Show that in this case all the ladder diagrams can be summed and obtain the resulting expression for Γ.

(d) Show that Γ diverges at a temperature when

$$\frac{1}{|V_0|} = \int \frac{d^3k}{(2\pi)^3}\frac{\tanh(\beta(\epsilon_k - \mu)/2)}{2(\epsilon_k - \mu)}.$$

This equation, first obtained by Bardeen, Cooper and Schrieffer, determines the superconducting transition temperature.