Intermediate Mechanics - PHY3221 Name:______________________________
Midterm II — Apr. 11, 2005

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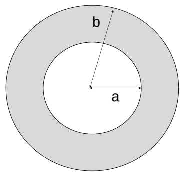


1. (10 pts) A thick spherical shell with uniform density has total mass M, inner radius a and outer radius b (see figure). Let r be the distance from the center of the shell.

(a) What is the gravitational field due to the shell when $r > b$?

The essential idea for this problem is that the gravitational field $\vec{g}$ at a distance r from the center of a spherically symmetric mass distribution is given by $\vec{g} = -(GM_{inside}/r^2)\hat{e}_r$ where M_{inside} is the total mass inside a sphere of radius r.

For Part (a), all the mass of the shell is inside this sphere, and so

$$\vec{g} = -\frac{GM}{r^2}\hat{\mathbf{e}}_r$$

(b) What is the gravitational field due to the shell when $a < r < b$?

To find M_{inside} for this case note that the total volume of the shell is $V = \frac{4}{3}\pi(b^3 - a^3)$, so the density is

$$\rho = \frac{M}{\frac{4}{3}\pi(b^3 - a^3)},$$

and, since the volume of the shell that's inside a sphere of radius r is $\frac{4}{3}\pi(r^3 - a^3)$,

$$M_{inside} = \frac{4}{3}\pi(r^3 - a^3)\frac{M}{\frac{4}{3}\pi(b^3 - a^3)} = M\frac{r^3 - a^3}{b^3 - a^3}.$$

Thus

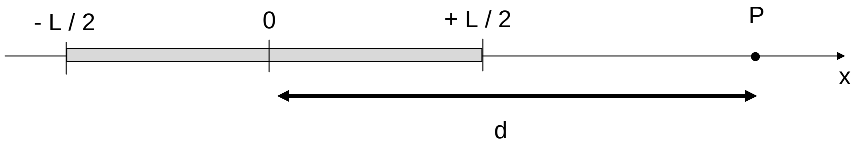
$$\vec{g} = -\frac{GM}{r^2}\frac{r^3 - a^3}{b^3 - a^3}\hat{\mathbf{e}}_r = -\frac{GM}{b^3 - a^3}\left(r - \frac{a^3}{r^2}\right)\hat{\mathbf{e}}_r$$

(c) What is the gravitational field due to the shell when $r < a$?

In this case $M_{inside} = 0$ and

$$\vec{g} = 0$$

2. (10 pts) Consider a thin rod of mass M and length L. The point P lies on the same axis as the rod at a distance d from the center of the rod, (see figure).



(a) Find the gravitational potential due to the rod at the point P.

The potential at the point P is

$$\Phi = -G \int \frac{dm}{r}$$

where

$$dm = \frac{M}{L}\, dx \qquad \textit{and} \qquad r = d - x.$$

Thus,

$$\Phi = -G \int \frac{dm}{r} = -G \int_{-L/2}^{L/2} \frac{M}{L} \frac{dx}{d-x} = \frac{GM}{L} \ln \left| \frac{d - L/2}{d + L/2} \right| = \frac{GM}{L} \ln \left| \frac{2d - L}{2d + L} \right| .$$

(b) Find the gravitational field due to the rod at the point P.

To find the gravitational field we take the gradient of Φ*, with the result*

$$\vec{g} = -\vec{\nabla}\Phi = -\frac{d\Phi}{dd}\hat{\mathbf{e}}_x = \frac{GM}{L}\left(\frac{-2}{L-2d} - \frac{2}{L+2d}\right)\hat{\mathbf{e}}_x$$

which can be simplified to give

$$\vec{g} = -\frac{4GM}{4d^2 - L^2}\,\hat{\mathbf{e}}_x$$

(c) In the limit that $d \gg L$ what is the leading contribution to the gravitational field due to the rod at the point P?

For $d \gg L$,

$$\vec{g} \simeq -\frac{GM}{d^2}\,\hat{\mathbf{e}}_x$$

3. (10 pts) Pluto orbits the sun in an elliptical orbit with eccentricity $\varepsilon = 0.25$. The distance of Pluto from the sun at perihelion (closest approach) is 4.4×10^{12} m. Newton's constant is $G = 6.67 \times 10^{-11}$ N m^2/kg^2 and the mass of the sun is 2.0×10^{30} kg.

(a) What is the distance of Pluto from the sun at aphelion (the point in the orbit which is furthest from the sun)?

Pluto's orbit is a conic section described by the equation,

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta \Rightarrow r = \frac{\alpha}{1 + \varepsilon \cos\theta}$$

from which it follows that distances of Pluto from the sun at perihelion (r_{min}) and aphelion (r_{max}) are

$$r_{min} = \frac{\alpha}{1+\varepsilon}; \quad r_{max} = \frac{\alpha}{1-\varepsilon}.$$

Thus we have

$$r_{max} = \frac{1+\epsilon}{1-\epsilon} r_{min} = \frac{1+0.25}{1-0.25} 4.4 \times 10^{12} \text{ m} = 7.3 \times 10^{12} \text{ m}$$

(b) What is the period of Pluto's orbit?

The easiest way to do this problem is to use the fact that Kepler's third law implies $\tau^2 \propto a^3$. Thus, if τ_p and τ_e are the orbtial periods, and a_p and a_e are the semimajor axes, of Pluto and the Earth, respectively, then

$$\frac{\tau_p}{\tau_e} = \left(\frac{a_p}{a_e}\right)^{3/2}$$

We know that

$$a_p = \frac{r_{max} + r_{min}}{2} = 5.85 \times 10^{12} \ m$$

and

$$a_e = 1.5 \times 10^{11} \ m$$

(This is the radius of earth's orbit, which is equal to the semimajor axis because the orbit is approximately circular). Thus we have

$$\frac{\tau_p}{\tau_e} = \left(\frac{5.85 \times 10^{12}}{1.5 \times 10^{11}}\right)^{3/2} \simeq 240$$

and

$$\tau_p \simeq 240 \text{ years}$$

(c) What is the velocity of Pluto at perihelion? (Hint: Use the fact that the total energy of an elliptical orbit depends only on its semimajor axis.)

The total energy of an orbit with semimajor axis a is $E = -k/(2a)$. Thus, if v is the velocity at perihelion, we have,

$$E = -\frac{k}{2a} = -\frac{k}{r_{min} + r_{max}} = \frac{1}{2}mv^2 - \frac{k}{r_{min}}$$

from which we find

$$\frac{1}{2}mv^2 = k\left(\frac{r_{min}}{r_{min}(r_{min} + r_{max})} - \frac{r_{min} + r_{max}}{r_{min}(r_{min} + r_{max})}\right) = k\frac{r_{max}}{r_{min}(r_{min} + r_{max})}$$

and, finally,

$$v = \sqrt{\frac{2k}{m}\frac{r_{max}}{r_{min}(r_{min} + r_{max})}}$$

and since $k = GMm$, where M is the mass of the sun,

$$\begin{aligned} v &= \sqrt{2GM\frac{r_{max}}{r_{min}(r_{min} + r_{max})}} \\ &= \sqrt{2(6.67 \times 10^{-11} Nm^2/kg^2)(2.0 \times 10^{30}\ kg)\frac{7.3 \times 10^{12}\ m}{(4.4 \times 10^{12}\ m)(11.7 \times 10^{12}\ m)}} \\ &= 6.2 \times 10^3\ m/s \end{aligned}$$