Physics 5524

Statistical Mechanics

Problem Set 10

Due: Wednesday, Apr. 2

10.1 In 1995 Bose-Einstein condensation was realized experimentally for dilute gases of alkali atoms (Li, Na, K, Rb, Cs). Unlike He^4, these gases are sufficiently dilute to be well described by the ideal Bose gas model.

In these experiments, atoms are trapped using inhomogeneous magnetic fields which couple to the atoms through the Zeeman coupling to the unpaired s-electron in the outermost shell. The resulting "magnetic trap" can be modeled as a three-dimensional harmonic potential $V(r) = \frac{1}{2}m\omega_0^2 r^2$ for which the single-particle energy levels are given by $\mathcal{E}'_{n_x,n_y,n_z} = \hbar\omega_0(n_x + n_y + n_z + 3/2)$. In what follows it is convenient to subtract off the zero-point energy so that the ground state has energy $\mathcal{E} = 0$ rather than $\frac{3}{2}\hbar\omega_0$. The energy levels are then

$$\mathcal{E}_{n_x,n_y,n_z} = \hbar\omega_0(n_x + n_y + n_z).$$

The total number of bosons N for such a harmonically confined gas is given by the usual sum over Bose occupation factors,

$$N = \sum_{n_x,n_y,n_z} n(\mathcal{E}_{n_x,n_y,n_z}) = \sum_{n_x,n_y,n_z} \frac{1}{z^{-1}e^{\beta\mathcal{E}_{n_x,n_y,n_z}} - 1} \tag{1}$$

where z is the fugacity.

(a) Show that if $k_BT \gg \hbar\omega_0$, the sum in Eq. (1) can be approximated by an integral (once one has properly taken into account the possibility of a macroscopic occupation, N_0, of the one-particle ground state).

(b) Using the result of (a), obtain an expression for the Bose-Einstein condensation temperature T_c for this gas. Show that if $N \gg 1$ the assumption that $k_BT_c \gg \hbar\omega_0$ is justified.

(c) Determine the temperature dependence of the condensate fraction, N_0/N, when $T \leq T_c$ for this gas.

(d) For $T < T_c$ obtain an expression for the energy of this gas as a function of N, ω_0 and T. From this determine the temperature dependence of the specific heat for $T < T_c$.

(e) For a trap with angular frequency $\omega_0 = 2\pi \times 100 s^{-1}$ (a typical number) and 10^6 atoms calculate T_c. (Don't be surprised if this temperature is *very* low!)

Some useful integrals:

$$\int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{1}{e^{x+y+z} - 1} = \zeta(3), \quad \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{x+y+z}{e^{x+y+z} - 1} = 3\zeta(4),$$

where $\zeta(n)$ is the Riemann zeta function.

10.2 Consider a 2-dimensional Debye solid of area A consisting of N ions. For each wave vector $\vec{k}$ there will then be 1 longitudinal mode and 1 transverse modes and the total number of normal modes will be $2N$. For a Debye solid the dispersion of these modes is taken to be $\omega_s(\vec{k}) = c|\vec{k}|$.

(a) Obtain an expression for the Debye wave vector k_D of this solid for a given ion density N/A. Also obtain expressions for the Debye frequency ω_D and Debye temperature Θ_D for this solid.

(b) Obtain an expression for $C_V(T)$ for this solid.

(c) Show in the low temperature limit ($T \ll \Theta_D$) that $C_V(T) \propto T^2$ and determine the prefactor.

(d) Show in the high temperature limit ($T \gg \Theta_D$) that $C_V(T) \simeq 2Nk_B$ and comment on the significance of this result.

Yet another useful integral,

$$\int_0^\infty dx \frac{x^3 e^x}{(e^x - 1)^2} = 6\zeta(3).$$

10.3 Pathria, Problem 7.27, Page 192.