**Physics 5524**

**Statistical Mechanics**

**Problem Set 11**

Due: Wednesday, Apr. 9

11.1 Consider a gas of photons in thermal equilibrium with the walls of a cavity of volume $V$ at temperature $T$.

(a) Write down the grand partition function $\mathcal{Z}(V,T)$ for this gas. (Remember that for a gas of photons $\mu = 0$ and so $z = 1$).

(b) Find an expression for the grand potential $\Sigma$ for this gas. Express your result as a function of $T$, $V$ and an integral over frequency $\omega$. (You need not perform the integral).

(c) Find an expression for the total energy $E$ of this gas, again expressed as a function of $T$, $V$ and an integral over frequency $\omega$.

(d) Using the results of Parts (b) & (c) show that $PV = \frac{1}{3}E$ for this gas. (**Hint**: To do this you will need to integrate by parts.)

11.2 (From the Spring 2004 Comprehensive Exam)
Consider a degenerate, ultra-relativistic ($mc^2 \ll cp_F$) gas of non-interacting electrons. In this limit, the energy of an electron is related to its momentum by $\mathcal{E}(\vec{p}) = c|\vec{p}|$. Consider $N$ such electrons in a volume $V$.

(a) At zero temperature, Find the chemical potential $\mu$ and the Fermi momentum $p_F$ for this gas as a function of $N$ and $V$.

(b) Find the total energy of the system $E$ at zero temperature as a function of $N$ and $V$.

(c) Use the fact that at $T = 0$ the pressure is given $P = -\frac{\partial E}{\partial V}$ to obtain an expression for $P$ and show that $PV = \frac{1}{3}E$ when $T = 0$.

(d) Write down the grand partition function for this gas at finite temperature, and use it to obtain an expression for the grand potential $\Sigma$. Express $\Sigma$ as a function of $T$, $V$, $\mu$ and an integral over energy. (Do not try to do the integral.)

(e) Find an expression for the total energy $E$ for this gas. Again, your answer should be expressed as a function of $T$, $V$, $\mu$ and an integral over energy. (Again, do not try to do the integral.)

(f) Using the results of Parts (d) and (e) show that $PV = \frac{1}{3}E$ for any temperature $T$ (not just $T = 0$). As in Prob. 11.1, to do this you will have to perform an integration by parts.

**Note:** Problems 11.1 and 11.2 illustrate a generic feature of ideal gases which holds regardless of whether they are described by Bose-Einstein, Fermi-Dirac or Maxwell-Boltzmann statistics. For an ideal gas in $D$ dimensions, if the energy dispersion of the particles is $\mathcal{E}(\vec{k}) = A|\vec{k}|^s$ then $PV = \frac{s}{D}E$.

11.3 Consider an ideal gas of $N$ spin-1/2 fermions of mass $m$ in *two* space dimensions. Let $A$ be the area of the gas so that the particle density is $n = N/A$.

(a) Determine the Fermi wave vector $k_F$, the Fermi energy $E_F$ and the Fermi temperature $T_F$ for this gas.

(b) Determine the density of states $a(\mathcal{E})$ for this gas.

(c) Obtain a closed form expression for the chemical potential $\mu$ for this gas as a function of temperature. Show that in the limit of high temperature ($T \gg T_F$) the fugacity is approximately given by $z \simeq \lambda^2/l^2$ where $\lambda$ is the thermal wave length and $l = \sqrt{A/N}$ is the mean particle spacing.

(d) Obtain the specific heat at constant area of this gas in both the low temperature $T \ll T_F$ and high temperature $T \gg T_F$ limits.