Physics 5524

Statistical Mechanics

Problem Set 12

Due: Wednesday, Apr. 16

11.3 Cont'd.

In Problem 11.3 you (hopefully) found that the one-particle density of states for a 2D nonrelativistic gas of spin-1/2 fermions confined to an area A is given by,

$$a(\mathcal{E}) = A\frac{m}{\pi\hbar^2}, \quad \mathcal{E} > 0,$$

($a(\mathcal{E}) = 0$ for $\mathcal{E} < 0$).

(d) Obtain the specific heat at constant area of this gas in both the low temperature $T \ll T_F$ and high temperature $T >> T_F$ limits.

12.1 Consider a noninteracting gas of ultrarelativistic spin-1/2 fermions with dispersion relation $\mathcal{E}(\vec{k}) = \hbar c|\vec{k}|$. The gas consists of N particles in a three-dimensional volume V and so has number density $n = N/V$.

(a) Obtain the one-particle density of states $a(\mathcal{E})$ for these fermions.

(b) Obtain an expression for the Fermi energy $\mathcal{E}_F$ of this gas as a function of the density n.

(c) Show that at $T = 0$ the total energy of this gas is $E = \frac{3}{4}N\mathcal{E}_F$.

(d) Use the fact that for an ultrarelativistic gas $PV = \frac{1}{3}E$ (which you proved in the last problem set) to show that at $T = 0$ the pressure is related to the density by $P \propto n^\alpha$ and determine the exponent α.

(e) At finite temperature, use the Sommerfeld expansion to obtain an expression for the chemical potential μ as a function of N, T and V which is valid to second order in T.

(f) Again using the Sommerfeld expansion, obtain an expression for the total energy E as a function of N, T and V valid to second order in T.

(g) Using the results of Parts (e) and (f) obtain an expression for the specific heat at constant volume C_V valid for $k_BT \ll \mathcal{E}_F$.

12.2 Now consider a gas of noninteracting spin-1/2 fermions with energy dispersion given by the usual relativistic expression for a particle of mass m,

$$\mathcal{E}(\vec{k}) = \sqrt{(mc^2)^2 + (\hbar c|\vec{k}|)^2}.$$

The gas consists of N particles in a volume V and so has number density $n = N/V$.

(a) Find an expression for the Fermi wave vector k_F for this gas as a function of number density n.

(b) Show that at $T = 0$ the total energy of this gas is given by

$$E = \frac{V}{\pi^2}\frac{m^4c^5}{\hbar^3}\int_0^{x_F} x^2\sqrt{x^2+1}dx, \tag{1}$$

where $x_F = \hbar k_F/mc$.

(c) Consider the nonrelativistic limit $\hbar k_F \ll mc$ (i.e. $x_F \ll 1$) of Eq. (1). By Taylor expanding the integrand to fourth order in x obtain an expression for the total energy valid in this limit. Express your answer in terms of N and V and show that it has the form $E \simeq Nmc^2 + E_{NR}$ where E_{NR} is the energy density of a nonrelativisitc gas of spin-1/2 fermions.

(d) Now consider the ultrarelativistic limit $\hbar k_F \gg mc$ (i.e. $x_F \gg 1$). Obtain an expression for the total energy valid in this limit and show that it agrees with the result obtained in Part (c) of Problem 12.1.

(e) Using the fact that at $T = 0$ the pressure is given by $P = -\left(\frac{\partial E}{\partial V}\right)_N$ show that in the two limits considered in Parts (c) and (d) the pressure obeys the law $P \propto n^\alpha$ and determine the exponent α in these two limits. (The nonrelativisitc result should agree with the result obtained in class, and the ultrarelativistic result should agree with that found in Part (d) of Problem 12.1.)

12.3 Consider the same relativistic gas discussed in Problem 12.2.

(a) Show that the one-particle density of states for the fermions in this gas is given by

$$a(\mathcal{E}) = \frac{V}{\pi^2}\frac{\mathcal{E}\sqrt{\mathcal{E}^2 - (mc^2)^2}}{\hbar^3 c^3}, \qquad \mathcal{E} \geq mc^2.$$

(b) Using the Sommerfeld expansion, show that the low-temperature specific heat of this gas is given by

$$C_V = Nk_B\pi^2 \frac{\sqrt{x_F^2+1}}{x_F^2}\frac{k_BT}{mc^2}. \tag{2}$$

where, as in Problem 12.2, $x_F = \hbar k_F/mc$. Verify that in the limit $x_F \ll 1$ Eq. (2) gives the correct result for a nonrelativistic gas of spin-1/2 fermions derived in class and in the limit $x_F \gg 1$ it agrees with the result you obtained in Part (g) of Problem 12.1 for an ultrarelativistic gas.