**Physics 5524**

**Statistical Mechanics**

**Problem Set 2**

Due: Wednesday, Jan. 23

2.1 (This is a modified version of Problem 1.12 in Goodstein. It is also closely related to Problem 9 on the Spring 2008 qualifying exam.)

There is a model of the thermal behavior of crystalline solids, according to which each of the $N$ atoms of the solid behaves like three independent harmonic oscillators (associated with oscillations in the $x$, $y$ and $z$ directions). The $3N$ harmonic oscillators (which are on distinguishable sites) all have the same frequency $\omega_0$. Their possible energy levels are

$$\mathcal{E}_n = \hbar\omega_0 \left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \cdots$$

(a) Show that the partition function for this system can be expressed as

$$\mathcal{Q}_N(T) = \left(Q_1(T)\right)^{3N}$$

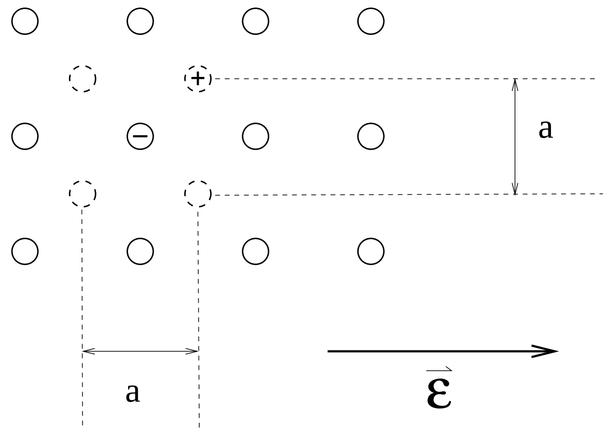
where $Q_1(T)$ is the partition function for a single oscillator. Obtain an expression for $Q_1(T)$.

(b) Using the result of (a), show that the Helmholtz free energy, $A(T, N)$, for this system is given by

$$A(T, N) = 3Nk_BT \ln\left(1 - e^{-\hbar\omega_0/k_BT}\right) + \frac{3}{2}N\hbar\omega_0. \tag{1}$$

(c) Obtain the heat capacity, $C(T) = T\left(\frac{\partial S}{\partial T}\right)_N$ for this system. Obtain approximate expressions for $C(T)$ valid in the low-temperature ($k_BT \ll \hbar\omega_0$) and high-temperature ($k_BT \gg \hbar\omega_0$) limits and sketch a plot of $C$ vs. $T$.

2.2 (From the Spring 1998 Comprehensive Exam). Consider a two dimensional solid which forms a square lattice as shown in the figure. The solid contains a dilute concentration $n$ of negatively charged impurity ions replacing some of the ordinary atoms of the solid. (If $A$ is the area of the solid then this means that the total number of impurities is $N = nA$.) Each negatively charged ion with charge $-e$ has in its vicinity a positive ion with charge $+e$. The positive ion is free to move so that in the absence of an external electric field it will be found with equal probability in any of the 4 interstitial sites surrounding the stationary negative ion as shown in the figure. Assume the density of ions is sufficiently small so that one may treat each negatively charged ion and its associated positive ion independently.




An electric field $\vec{\mathcal{E}}$ is applied to the solid in the $x$ direction as shown.

(a) Compute the partition function for this system.

(b) Compute the Helmoltz free energy for this system.

(c) Compute the specific heat for this system.

(d) Compute the induced dipole moment per unit area for this system and give approximate expressions for your result which are valid in the limit of very low and very high temperatures.