

## PHY5524 Problem Set 1: Solution

### Problem 1

(a) If  $n$  is the number of segments which point up and  $L$  is the length of the rubber band, then

$$\frac{L}{a} = N - 2n \Rightarrow n = \frac{N}{2} - \frac{L}{2a} \quad (1)$$

The number of microstates corresponding to length  $L$  is then  $N$  choose  $n$ , i.e.

$$\# \text{ of microstates} = \binom{N}{n} = \frac{N!}{(N-n)!n!} = \frac{N!}{\left(\frac{N}{2} + \frac{L}{2a}\right)! \left(\frac{N}{2} - \frac{L}{2a}\right)!} \quad (2)$$

From this, we find the entropy is

$$S = k_B \ln \frac{N!}{\left(\frac{N}{2} + \frac{L}{2a}\right)! \left(\frac{N}{2} - \frac{L}{2a}\right)!} \quad (3)$$

Or, since  $E = -mgL$ ,

$$S(E, N) = k_B \ln \frac{N!}{\left(\frac{N}{2} - \frac{E}{2mga}\right)! \left(\frac{N}{2} + \frac{E}{2mga}\right)!} \quad (4)$$

(b) For  $N \gg 1$ , if we apply Stirling's approximation ( $\ln N! = N \ln N - N + O(\ln N)$ ) we obtain

$$S \simeq k_B \left( N \ln N - \left(\frac{N}{2} + \frac{E}{2mga}\right) \ln \left(\frac{N}{2} + \frac{E}{2mga}\right) - \left(\frac{N}{2} - \frac{E}{2mga}\right) \ln \left(\frac{N}{2} - \frac{E}{2mga}\right) \right) \quad (5)$$

or, after simplifying things a bit,

$$S(E, N) = -k_B N \left( \left(\frac{1}{2} + \frac{E/N}{2mga}\right) \ln \left(\frac{1}{2} + \frac{E/N}{2mga}\right) + \left(\frac{1}{2} - \frac{E/N}{2mga}\right) \ln \left(\frac{1}{2} - \frac{E/N}{2mga}\right) \right) \quad (6)$$

(c) We can use the fact that  $\left(\frac{\partial S}{\partial E}\right)_N = \frac{1}{T}$  to obtain

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{k_B}{2mga} \ln \frac{mga - E/N}{mga + E/N} \quad (7)$$

We can then solve for  $E$  by exponentiating both sides

$$e^{2mga/k_B T} = \frac{mga - E/N}{mga + E/N} \quad (8)$$

and doing a little algebra

$$(Nmga + E)e^{2mga/k_B T} = Nmga - E \quad (9)$$

$$E(1 + e^{2mga/k_B T}) = Nmga(1 - e^{2mga/k_B T}) \quad (10)$$

to obtain

$$E(N, T) = Nmga \frac{1 - e^{2mga/k_B T}}{1 + e^{2mga/k_B T}} = -Nmga \tanh \frac{mga}{k_B T} \quad (11)$$

(d) Since  $E = -mgL$ ,

$$L = -\frac{E}{mg} = +Na \tanh \frac{mga}{k_B T} \quad (12)$$

### Problem 2

We assume that  $S = f(\Omega)$  where  $f$  is some universal function. If we have two systems with  $\Omega_1$  and  $\Omega_2$ , then  $S_{tot} = S_1 + S_2 = f(\Omega_1) + f(\Omega_2)$ . At the same time, since the total number of microstates of the two systems taken together is  $\Omega_{tot} = \Omega_1\Omega_2$ ,  $S_{tot} = f(\Omega_1\Omega_2)$ .

It follows that the function  $f$  must satisfy the following equation

$$f(\Omega_1\Omega_2) = f(\Omega_1) + f(\Omega_2) \quad (13)$$

for any  $\Omega_1$  and  $\Omega_2$ . Clearly  $f(\Omega) \propto \ln \Omega$  satisfies this equation, but is this the only solution? It's easy to prove that it is.

If we differentiate both sides of (13) w.r.t.  $\Omega_2$  while keeping  $\Omega_1$  fixed we obtain

$$\Omega_1 f'(\Omega_1\Omega_2) = f'(\Omega_2) \quad (14)$$

Since this equation must hold for all  $\Omega_1$  and  $\Omega_2$ , it must hold when  $\Omega_2 = 1$ . So let's set  $\Omega_2 = 1$  and let  $k = f'(1)$ . This implies that

$$f'(\Omega_1) = \frac{k}{\Omega_1} \quad (15)$$

Integrating this w.r.t.  $\Omega_1$  then yields

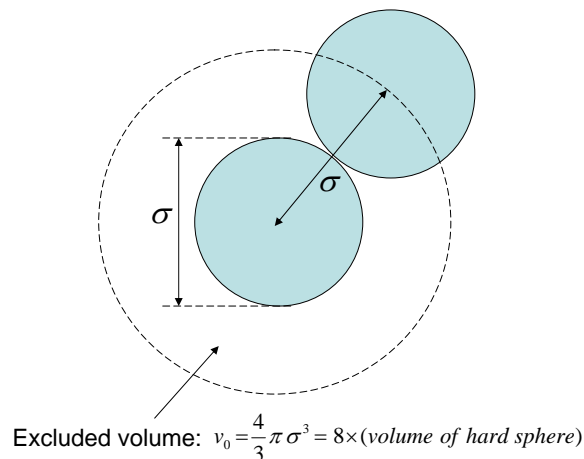
$$f(\Omega_1) = k \ln \Omega_1 + C \quad (16)$$

where  $C$  is an integration constant. Plugging this into the starting expression, we see that the integration constant  $C$  must equal 0. Thus we find that not only is  $f(\Omega) = k \ln(\Omega)$  a solution to (13), it is the *only* solution.

### Problem 3

In this problem we consider a classical gas of hard spheres with diameter  $\sigma$ . Each such hard sphere excludes a volume  $v_0$  from each of the other hard spheres in the gas. (Note that since we are assuming that  $Nv_0 \ll V$  we are in the *dilute* limit, so in calculating the excluded volume due to a given set of hard spheres, we can assume these spheres are well separated so each excludes the same volume  $v_0$ .)

The following figure makes it clear that  $v_0$  is 8 times larger than the volume of a given hard sphere.



We expect the number of microstates for a given sphere to be proportional to the volume of space it can occupy. If the first sphere can occupy a volume  $V$ , then the second sphere occupies a volume  $V - v_0$ , the third a volume  $V - 2v_0$ , and so on. Thus we expect that the  $V$  dependence of  $\Omega$  takes the following form

$$\Omega \propto V(V - v_0)(V - 2v_0) \cdots (V - (N - 1)v_0) \quad (17)$$

We can then use the fact that

$$\left(\frac{\partial \ln \Omega}{\partial V}\right)_{E,N} = \frac{P}{k_B T} \quad (18)$$

to obtain

$$\frac{P}{k_B T} = \frac{\partial \ln \Omega}{\partial V} = \sum_{n=0}^{N-1} \frac{1}{V - nv_0} \quad (19)$$

Now, since we are assuming that  $Nv_0 \ll V$  we are justified in Taylor expanding each term in the sum on the right hand side to obtain

$$\frac{P}{k_B T} \simeq \sum_{n=0}^{N-1} \left( \frac{1}{V} + n \frac{v_0}{V^2} \right) \quad (20)$$

Using the fact that

$$\sum_{n=1}^{N-1} n = \frac{N(N-1)}{2} \quad (21)$$

we then have

$$\frac{P}{k_B T} \simeq \frac{N}{V} + \frac{N(N-1)}{2} \frac{v_0}{V^2} \quad (22)$$

$$= \frac{N}{V} \left( 1 + \frac{N-1}{2} \frac{v_0}{V} \right) \quad (23)$$

$$\simeq \frac{N}{V - (N-1)v_0/2} \quad (24)$$

$$\simeq \frac{N}{V - Nv_0/2}. \quad (25)$$

Since  $v_0$  is eight times the volume of each hard sphere,  $Nv_0/2$  is *four* times the total volume taking up by the  $N$  spheres forming the gas. Our equation of state is then

$$P(V - b) = Nk_B T \quad (26)$$

where  $b$  is four times the volume taking up the particles.