

PHY5524 Problem Set 12: Solution

Problem 3 from Set# 11 (Cont'd)

(d) To determine the specific heat we can use the Sommerfeld expansion,

$$\int_0^\infty \phi(\mathcal{E}) \frac{1}{z^{-1}e^{\beta\mathcal{E}} + 1} d\mathcal{E} = \int_0^\mu \phi(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6} (k_B T)^2 \left. \frac{d\phi}{d\mathcal{E}} \right|_{\mathcal{E}=\mu} + O(T^4) \quad (1)$$

First, we determine the chemical potential by fixing the number of particles to be N ,

$$N = \int_0^\infty a(\mathcal{E}) \frac{1}{z^{-1}e^{\beta\mathcal{E}} + 1} d\mathcal{E} = \int_0^\mu a(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6} (k_B T)^2 a'(\mu) = \mu A \frac{m}{\pi \hbar^2} \quad (2)$$

Here we have used the fact that since $a(\mathcal{E})$ is constant, a' (and all subsequent derivatives) vanish. This Eq. (2) is the *exact* result for the Sommerfeld expansion, valid to all orders (however, see the next paragraph). Solving for μ we find

$$\mu(T) = \mathcal{E}_F = \frac{\pi \hbar^2}{m} n. \quad (3)$$

It is instructive to compare this with the exact result obtained in Part (c) of this problem,

$$\mu(T) = \mathcal{E}_F + k_B T \ln(1 - e^{-T_F/T}) \quad (4)$$

For $T \ll T_F$, $e^{-T_F/T} \ll 1$ and the \ln can be Taylor expanded, with the result

$$\mu(T) = \mathcal{E}_F - k_B T e^{-T_F/T} \quad (5)$$

Thus we see that the Sommerfeld expansion result Eq. (2) *misses* the term $k_B T e^{-T_F/T}$. The reason for this is that this term is an *essential singularity*, i.e. it has no Taylor expansion in powers of T . The fact that the Sommerfeld expansion misses such nonanalytic terms is of no significance, because when $T \ll T_F$ the exponential factor $e^{-T_F/T}$ is *extremely* small and can be safely ignored.

For the total energy we then have

$$E = \int_0^\infty a(\mathcal{E}) \mathcal{E} \frac{1}{z^{-1}e^{\beta\mathcal{E}} + 1} d\mathcal{E} = \int_0^\mu a(\mathcal{E}) \mathcal{E} d\mathcal{E} + \frac{\pi^2}{6} (k_B T)^2 (\mu a'(\mu) + a(\mu)) + O(T^4) \quad (6)$$

For the 2D gas considered here, $a(\mathcal{E})$ is constant and, as shown above, in the limit $T \ll T_F$ we can take $\mu = \mathcal{E}_F$ (in 2D there is no order T^2 contribution to the chemical potential), thus we have

$$E(T) = \int_0^{\mathcal{E}_F} a(\mathcal{E}) \mathcal{E} d\mathcal{E} + \frac{\pi^2}{6} (k_B T)^2 a(\mathcal{E}_F) \quad (7)$$

$$= E(0) + \frac{\pi^2}{6} (k_B T)^2 a(\mathcal{E}_F) \quad (8)$$

Finally, we find that the specific heat is

$$C_V = \left(\frac{\partial E}{\partial T} \right)_{N,A} = \frac{\pi^2}{3} k_B^2 a(\mathcal{E}_F) T \quad (9)$$

Note that in 2D $a(\mathcal{E}_F) = N/\mathcal{E}_F$ so the specific heat can also be expressed as

$$C_V = \frac{\pi^2}{3} N k_B \frac{k_B T}{\mathcal{E}_F} \quad (10)$$

In the high temperature limit, you showed in Part (c) of this problem that

$$z \simeq \frac{1}{2} \frac{\lambda^2}{l^2} = \frac{1}{2} \left(\frac{h^2}{2\pi m k_B T} \right) \frac{N}{A} \ll 1 \quad (11)$$

In this limit, since $z^{-1} \gg 1$, the Fermi occupation factor can be approximated by the Maxwell-Boltzmann limit and we find

$$E = \int_0^\infty a(\mathcal{E}) \mathcal{E} \frac{1}{z^{-1} e^{\beta \mathcal{E}} + 1} d\mathcal{E} \quad (12)$$

$$\simeq z \int_0^\infty A \frac{m}{\pi \hbar^2} \mathcal{E} e^{-\beta \mathcal{E}} d\mathcal{E} \quad (13)$$

$$= \frac{1}{2} \left(\frac{\hbar^2}{2\pi m k_B T} \right) \frac{N}{A} A \frac{m}{\pi \hbar^2} (k_B T)^2 \quad (14)$$

$$= N k_B T \quad (15)$$

The specific heat for $T \gg T_F$ is then

$$C_V = \left(\frac{\partial E}{\partial T} \right)_N = N k_B \quad (16)$$

which is the expected equipartition result for a 2 dimensional classical monatomic gas.

Problem 1.

(a) The density of states for ultrarelativistic spin-1/2 particles is

$$a(\mathcal{E}) = \sum_{\vec{k}, s} \delta(\mathcal{E} - \hbar c |\vec{k}|) \quad (17)$$

$$= 2V \int \frac{d^3 k}{(2\pi)^3} \delta(\mathcal{E} - \hbar c |\vec{k}|) \quad (18)$$

$$= 2V \frac{1}{(2\pi)^3} 4\pi \int_0^\infty k^2 dk \delta(\mathcal{E} - \hbar c k) \quad (19)$$

$$= 2V \frac{1}{(2\pi)^3} 4\pi \left(\frac{\mathcal{E}}{\hbar c} \right)^2 \frac{1}{\hbar c} \quad (20)$$

$$= \frac{V}{\pi^2} \frac{\mathcal{E}^2}{\hbar^3 c^3} \quad (21)$$

(b) It is convenient to first find the Fermi wave vector in the usual way,

$$N = 2V \int_{|\vec{k}| < k_F} \frac{d^3 k}{(2\pi)^3} = 2 \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_F^3 = V \frac{k_F^3}{3\pi^2} \quad (22)$$

(The factor of 2 accounts for the two spin states). From this we find

$$k_F = (3\pi^2 n)^{1/3} \quad (23)$$

where $n = N/V$. Note that this result is independent of the dispersion (it only depends on dimensionality).

The Fermi energy is then

$$\mathcal{E}_F = \hbar c k_F = \hbar c (3\pi^2 n)^{1/3} \quad (24)$$

Alternatively, we can use the density of states found in (a)

$$N = \int_0^{\mathcal{E}_F} a(\mathcal{E}) d\mathcal{E} = V \frac{1}{\pi^2} \frac{1}{\hbar^3 c^3} \int_0^{\mathcal{E}_F} \mathcal{E}^2 d\mathcal{E} = V \frac{1}{\pi^2} \frac{1}{\hbar^3 c^3} \frac{1}{3} \mathcal{E}_F^3 \quad (25)$$

Solving for \mathcal{E}_F then gives

$$\mathcal{E}_F = \hbar c (3\pi^2 n)^{1/3} \quad (26)$$

as above.

(c) We can do this part just using the fact that the density of states goes as $a(\mathcal{E}) = C \mathcal{E}^2$ for some constant C .

For this density of states, the number of particles N is

$$N = \int_0^{\mathcal{E}_F} C\mathcal{E}^2 d\mathcal{E} = C\frac{1}{3}\mathcal{E}_F^3 \quad (27)$$

and the total energy is

$$E = \int_0^{\mathcal{E}_F} C\mathcal{E}^3 d\mathcal{E} = C\frac{1}{4}\mathcal{E}_F^4 \quad (28)$$

Dividing Eq. (28) by Eq. (27) we find

$$\frac{E}{N} = \frac{3}{4}\mathcal{E}_F \quad (29)$$

or, equivalently,

$$E = \frac{3}{4}N\mathcal{E}_F \quad (30)$$

(d) Given that we have already proven (in the previous problem set) that $PV = \frac{1}{3}E$ for this gas, we find here that

$$PV = \frac{1}{3}E = \frac{1}{4}N\mathcal{E}_F = \frac{1}{4}N\hbar c(3\pi^2 n)^{1/3} \quad (31)$$

and so

$$P = \frac{(3\pi^2)^{1/3}\hbar c}{4}n^{4/3}. \quad (32)$$

(e) To find the chemical potential we apply the Sommerfeld expansion (see Eq. (1) above) to the total number of particles

$$N = \int_0^{\infty} a(\mathcal{E}) \frac{1}{z^{-1}e^{\beta\mathcal{E}} + 1} d\mathcal{E} \quad (33)$$

$$= \int_0^{\mu} a(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6}(k_B T)^2 a'(\mu) + O(T^4) \quad (34)$$

$$= \int_0^{\mathcal{E}_F} a(\mathcal{E}) d\mathcal{E} + \int_{\mu}^{\mathcal{E}_F} a(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6}(k_B T)^2 a'(\mu) + O(T^4) \quad (35)$$

$$= \int_0^{\mathcal{E}_F} a(\mathcal{E}) d\mathcal{E} + (\mu - \mathcal{E}_F)a(\mathcal{E}_F) + \frac{\pi^2}{6}(k_B T)^2 a'(\mathcal{E}_F) + O(T^4) \quad (36)$$

where in the last equality we have used the fact that $\mu - \mathcal{E}_F \sim (k_B T)^2/\mathcal{E}_F$ to replace the integral from μ to \mathcal{E}_F of $a(\mathcal{E})$ by $(\mu - \mathcal{E}_F)a(\mathcal{E}_F)$ and to replace μ by \mathcal{E}_F in the $O((k_B T)^2)$ term. The errors we make in doing this are all of order T^4 and can be ignored. Thus we have

$$N = N + (\mu - \mathcal{E}_F)a(\mathcal{E}_F) + \frac{\pi^2}{6}(k_B T)^2 a'(\mathcal{E}_F) \quad (37)$$

which, after solving for μ , gives

$$\mu = \mathcal{E}_F - \frac{\pi^2}{6}(k_B T)^2 \frac{a'(\mathcal{E}_F)}{a(\mathcal{E}_F)} + O(T^4) \quad (38)$$

Since for ultrarelativistic particles the density of states goes as $\sim \mathcal{E}^2$ we have

$$a(\mathcal{E}) = C\mathcal{E}^2; \quad a'(\mathcal{E}) = 2C\mathcal{E} \quad (39)$$

and so

$$\frac{a'(\mathcal{E}_F)}{a(\mathcal{E}_F)} = \frac{2}{\mathcal{E}_F} \quad (40)$$

Thus we find

$$\mu = \mathcal{E}_F \left(1 - \frac{\pi^2}{3} \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 \right) \quad (41)$$

Finally, using the fact that $\mathcal{E}_F = \hbar c (3\pi^2 n)^{1/3}$ we can express μ as a function of N , V and T .

$$\mu = \hbar c \left(3\pi^2 \frac{N}{V} \right)^{1/3} \left(1 - \frac{\pi^2}{3} \left(\frac{k_B T}{(3\pi^2 N/V)^{1/3}} \right)^2 \right) \quad (42)$$

(f) Again using the Sommerfeld expansion, we find that to order T^2 the total energy is

$$E = \int_0^\infty a(\mathcal{E}) \mathcal{E} \frac{1}{z^{-1} e^{\beta \mathcal{E}} + 1} d\mathcal{E} = \int_0^\mu \mathcal{E} a(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6} (k_B T)^2 (\mu a'(\mu) + a(\mu)) + O(T^4) \quad (43)$$

As in Part (e), we break the integral from 0 to μ in the first term of the expansion into an integral from 0 to \mathcal{E}_F and an integral from \mathcal{E}_F to μ ,

$$E = \int_0^{\mathcal{E}_F} \mathcal{E} a(\mathcal{E}) d\mathcal{E} + \int_{\mathcal{E}_F}^\mu \mathcal{E} a(\mathcal{E}) d\mathcal{E} + \frac{\pi^2}{6} (k_B T)^2 (\mu a'(\mu) + a(\mu)) + O(T^4) \quad (44)$$

The first term on the RHS above is then the total energy at $T = 0$. Making the same approximations as in Part (e) on the remaining terms we find

$$E(T) = E(0) + (\mu - \mathcal{E}_F) \mathcal{E}_F a(\mathcal{E}_F) + \frac{\pi^2}{6} (k_B T)^2 (\mathcal{E}_F a'(\mathcal{E}_F) + a(\mathcal{E}_F)) + O(T^4) \quad (45)$$

Plugging in the expression for $\mu - \mathcal{E}_F$ derived in Part (e) we then find that

$$E(T) = E(0) + \frac{\pi^2}{6} (k_B T)^2 a(\mathcal{E}_F) \quad (46)$$

Again using the fact that $a(\mathcal{E}) = C\mathcal{E}^2$ we have

$$N = \int_0^{\mathcal{E}_F} C\mathcal{E}^2 d\mathcal{E} = C \frac{1}{3} \mathcal{E}_F^3 \quad (47)$$

from which we can determine the coefficient C ,

$$C = \frac{3N}{\mathcal{E}_F^3} \quad (48)$$

Thus the density of states can be written

$$a(\mathcal{E}) = \frac{3N}{\mathcal{E}_F^3} \mathcal{E}^2 \quad (49)$$

and the density of states at the Fermi energy is

$$a(\mathcal{E}_F) = \frac{3N}{\mathcal{E}_F} \quad (50)$$

Using the expression derived in Part (c) for $E(T = 0)$ we then have

$$E(T) = \frac{3}{4} N \mathcal{E}_F + \frac{\pi^2}{6} (k_B T)^2 \frac{3N}{\mathcal{E}_F} \quad (51)$$

Given that $\mathcal{E}_F = \hbar c (3\pi^2 N/V)^{1/3}$ we then have E as a function of T , V and N .

(g) The specific heat is

$$C_V = \left(\frac{\partial E}{\partial T} \right)_{V,N} = \pi^2 N k_B \frac{k_B T}{\mathcal{E}_F} \quad (52)$$

Problem 2.

(a) We determine the Fermi wave vector in exactly the same way we did in Part (b) of the previous problem,

$$N = 2 \frac{V}{(2\pi)^3} \frac{4}{3} \pi k_F^3 \quad (53)$$

$$\Rightarrow k_F = (3\pi^2 n)^{1/3} \quad (54)$$

where $n = N/V$. Note that this is the same result obtained in Part (b) of the previous problem, reflecting the fact that the Fermi wave vector is independent of the dispersion of the particles.

(b) At $T = 0$ only states with $|\vec{k}| < k_F$ are occupied. Thus the total energy is

$$E = 2V \int_{|\vec{k}| < k_F} \frac{d^3k}{(2\pi)^3} \sqrt{(mc^2)^2 + (\hbar c|\vec{k}|)^2} \quad (55)$$

$$= 2V \frac{1}{(2\pi)^3} 4\pi \int_0^{k_F} k^2 dk \sqrt{(mc^2)^2 + (\hbar ck)^2} \quad (56)$$

$$= \frac{V}{\pi^2} mc^2 \int_0^{k_F} k^2 dk \sqrt{1 + \left(\frac{\hbar k}{mc}\right)^2} \quad (57)$$

Making the change of variables, $x = \frac{\hbar k}{mc}$, we then find

$$E = \frac{V}{\pi^2} mc^2 \left(\frac{mc}{\hbar}\right)^3 \int_0^{x_F} x^2 \sqrt{1 + x^2} dx \quad (58)$$

$$= \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \int_0^{x_F} x^2 \sqrt{1 + x^2} dx \quad (59)$$

where $x_F = \hbar k_F / (mc)$

(c) In the nonrelativistic limit $p_F = \hbar k_F \ll mc$. Thus $x_F \ll 1$. In this limit we can Taylor expand the integrand to fourth order in x with the result

$$E \simeq \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \int_0^{x_F} x^2 \left(1 + \frac{1}{2} x^2\right) dx \quad (60)$$

$$= \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \left(\frac{x_F^3}{3} + \frac{1}{2} \frac{x_F^5}{5}\right) \quad (61)$$

$$= \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \left(\frac{\hbar^3 k_F^3}{3(mc)^3} + \frac{\hbar^5 k_F^5}{10(mc)^5}\right) \quad (62)$$

$$= mc^2 V \frac{k_F^3}{3\pi^2} + V \frac{k_F^3}{3\pi^2} \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} \quad (63)$$

$$= Nm c^2 + N \frac{3}{5} \mathcal{E}_F^{NR} \quad (64)$$

where we have used the fact that $N = V k_F^3 / (4\pi^2)$ and $\mathcal{E}_F^{NR} = \frac{\hbar^2 k_F^2}{2m}$ is the Fermi energy of a nonrelativistic gas.

(d) In the ultrarelativistic limit $p_F = \hbar k_F \gg mc$ and so $x_F \gg 1$. In this limit, we can safely ignore the 1 under the radical in the integrand. Thus we have

$$E \simeq \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \int_0^{x_F} x^3 dx = \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \frac{x_F^4}{4} \quad (65)$$

$$= \frac{V}{\pi^2} \frac{m^4 c^5}{\hbar^3} \frac{1}{4} \left(\frac{\hbar k_F}{mc}\right)^4 \quad (66)$$

$$= V \frac{k_F^3}{3\pi^2} \frac{3}{4} \hbar c k_F = N \frac{3}{4} \mathcal{E}_F^{UR} \quad (67)$$

where $\mathcal{E}_F^{UR} = \hbar c k_F$ is the Fermi energy of an ultrarelativistic gas.

(e) In the nonrelativistic limit, using the fact that $k_F = (3\pi^2 N/V)^{1/3}$ we have

$$E = Nmc^2 + N \frac{3}{5} \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3} = A \frac{N^{5/3}}{V^{2/3}} \quad (68)$$

It follows that in this limit the pressure depends on the density according to the following power law,

$$P = - \left(\frac{\partial E}{\partial V} \right)_N \propto n^{5/3} \quad (69)$$

In the ultrarelativistic limit, again using the fact that $k_F = (3\pi^2 N/V)^{1/3}$ we have

$$E = N \frac{3}{4} \hbar c k_F = N \frac{3}{4} \hbar c \left(3\pi^2 \frac{N}{V} \right)^{1/3} \quad (70)$$

and so in this limit the pressure depends on density according to a different power law,

$$P = - \left(\frac{\partial E}{\partial V} \right)_N \propto n^{4/3} \quad (71)$$

in agreement with the previous problem.

Problem 3

(a) The density of states for these relativistic particles is

$$a(\mathcal{E}) = \sum_{\vec{k}, s} \delta(\mathcal{E} - \mathcal{E}(\vec{k})) \quad (72)$$

$$= 2V \int \frac{d^3k}{(2\pi)^3} \delta(\mathcal{E} - \sqrt{(mc^2)^2 + (\hbar ck)^2}) \quad (73)$$

$$= 2V \frac{1}{(2\pi)^3} 4\pi \int_0^\infty k^2 dk \frac{\delta(k - \sqrt{\mathcal{E}^2 - (mc^2)^2}/\hbar c)}{\hbar^2 c^2 k ((mc^2)^2 + (\hbar ck)^2)^{-1/2}} \quad (74)$$

$$= \frac{V}{\pi^2} \frac{\mathcal{E} \sqrt{\mathcal{E}^2 - (mc^2)^2}}{\hbar^3 c^3}, \quad \mathcal{E} \geq mc^2. \quad (75)$$

Note that if $\mathcal{E} < mc^2$ the delta function is never satisfied and the density of states is zero.

(b) Using the Sommerfeld expansion as we did in class (see also Parts (e),(f) and (g) from Problem 1 of this HW set), one finds that, to order T^2 , the temperature dependence of the total energy of Fermi gas is

$$E(T) = E(0) + \frac{\pi^2}{6} (k_B T)^2 a(\mathcal{E}_F) \quad (76)$$

and the specific heat is

$$C_V = \left(\frac{\partial E}{\partial T} \right)_{N,V} = \frac{\pi^2}{3} k_B^2 T a(\mathcal{E}_F) \quad (77)$$

For the present problem, the Fermi energy is

$$\mathcal{E}_F = \sqrt{(mc^2)^2 + (\hbar ck_F)^2} = mc^2 \sqrt{1 + x_F^2} \quad (78)$$

and the density of states at the Fermi energy is

$$a(\mathcal{E}_F) = \frac{V}{\pi^2} \frac{\mathcal{E}_F \sqrt{\mathcal{E}_F^2 - (mc^2)^2}}{\hbar^3 c^3} \quad (79)$$

$$= \frac{V}{\pi^2} (mc^2)^2 \frac{x_F \sqrt{1 + x_F^2}}{\hbar^3 c^3} \quad (80)$$

$$= V \frac{k_F^3}{3\pi^2} 3 (mc^2)^2 \frac{x_F \sqrt{1 + x_F^2}}{\hbar^3 c^3 k_F^3} \quad (81)$$

$$= N 3 (mc^2)^2 \frac{x_F \sqrt{1 + x_F^2}}{c^3 x_F^3 (mc)^3} \quad (82)$$

$$= 3N \frac{\sqrt{x_F^2 + 1}}{x_F^2} \frac{1}{mc^2} \quad (83)$$

where $x_F = \hbar k_F/mc$, as in the previous problem. Thus we find that

$$C_V = Nk_B\pi^2 \frac{\sqrt{x_F^2 + 1}}{x_F} \frac{k_B T}{mc^2} \quad (84)$$

In the nonrelativistic limit, $x_F \ll 1$, we find

$$C_V \simeq Nk_B\pi^2 \frac{k_B T}{mc^2 x_F^2} = Nk_B\pi^2 \frac{k_B T}{\hbar^2 k_F^2/m} = Nk_B \frac{\pi^2}{2} \frac{k_B T}{\mathcal{E}_F^{NR}} \quad (85)$$

which agrees with the result derived in class.

In the ultrarelativistic limit, $x_F \gg 1$, we find

$$C_V \simeq Nk_B\pi^2 \frac{k_B T}{x_F mc^2} = Nk_B\pi^2 \frac{k_B T}{\hbar c k_F} = Nk_B\pi^2 \frac{k_B T}{\mathcal{E}_F^{UR}} \quad (86)$$

which agrees with Problem 1.