PHY5524
Problem Set 8: Solution

Problem 1

(a) The partition function for a single particle in a classical 3D gas is

$$Q_1^{(3D)} = \int \frac{d^3r d^3p}{h^3} e^{-\beta p^2/2m} = V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} \tag{1}$$

For N particles the partition function is then

$$Q_N^{(3D)} = \frac{1}{N!} \left(Q_1^{(3D)} \right)^N \tag{2}$$

(b) To find the chemical potential we first find the Helmholtz free energy

$$A^{(3D)} = -k_B T \ln Q_N^{(3D)} = k_B T N \left(\ln \frac{N}{Q_1^{(3D)}} - 1 \right) \tag{3}$$

(where we have used Stirling's approximation $\ln N! = N \ln N - N$. The chemical potential is then

$$\mu_{3D} = \left(\frac{\partial A}{\partial N} \right)_{T,V} = k_B T \ln \frac{N}{Q_1^{(3D)}} \tag{4}$$

where $N = N_1$, the number of particles in the 3D gas.

(c) The partition function for a single particle in a classical 2D gas with energy $E = p^2/2m - \eta$ is

$$Q_1^{(2D)} = \int \frac{d^2r d^2p}{h^2} e^{-\beta(p^2/2m - \eta)} = A \left(\frac{2\pi m k_B T}{h^2} \right)^2 e^{\beta\eta} \tag{5}$$

(Here A is the area of the gas — not to be confused with the free energy!) For N particles the partition function is then

$$Q_N^{(2D)} = \frac{1}{N!} \left(Q_1^{2D} \right)^N \tag{6}$$

and the Helmholtz free energy is

$$A^{(2D)} = -k_B T \ln Q_N^{(2D)} = k_B T N \left(\ln \frac{N}{Q_1^{2D}} - 1 \right) \tag{7}$$

and, finally, the chemical potential of the 2D gas is

$$\mu_{2D} = \left(\frac{\partial A}{\partial N} \right)_{T,V} = k_B T \ln \frac{N}{Q_1^{(2D)}} \tag{8}$$

where $N = N_2$, the number of particles in the 2D gas.

(d) If the 2D and 3D gases are in equilibrium, their chemical potentials must be equal. Thus we require

$$\mu_{3D} = \mu_{2D} \tag{9}$$

which gives

$$k_B T \ln \frac{N_1}{Q_1^{(3D)}} = k_B T \ln \frac{N_2}{Q_1^{(2D)}} \tag{10}$$

This in turn implies

$$\frac{N_1}{Q_1^{(3D)}} = \frac{N_2}{Q_1^{(2D)}} \tag{11}$$

which, after plugging in expressions for $Q^{(3D)}$ and $Q^{(2D)}$, yields

$$\frac{N_2}{A} = \frac{N_1}{V}\left(\frac{h^2}{2mk_BT}\right)^{1/2} e^{\eta/k_BT} \tag{12}$$

Finally, we can compute the pressure of the 3D gas as follows

$$P = -\left(\frac{\partial A^{(3D)}}{\partial V}\right)_{N,T} = N_1\frac{k_BT}{V} \tag{13}$$

This, of course, implies that the 3D gas satisfies the ideal gas law. Thus we have

$$\frac{N_1}{V} = \frac{P}{k_BT} \tag{14}$$

Putting everything together we obtain the following expression for the density of the 2D gas as a function of the pressure of the 3D gas,

$$n = \frac{N_2}{A} = \frac{P}{k_BT}\left(\frac{h^2}{2mk_BT}\right)^{1/2} e^{\eta/k_BT} \tag{15}$$

Problem 2

(a) For a 2D gas with dispersion $\epsilon(\vec{k}) = \hbar^2k^2/2m$, the density of states is given by

$$a(\mathcal{E}) = A\int\frac{d^2k}{(2\pi)^2}\delta\left(\mathcal{E} - \frac{\hbar^2k^2}{2m}\right) \tag{16}$$

where A is the area of the 2D gas. For $\mathcal{E} < 0$ the delta function is never satisfied and $a(\mathcal{E})$ is zero. For $\mathcal{E} > 0$ we evaluate $a(\mathcal{E})$ as follows

$$a(\mathcal{E}) = \frac{A}{(2\pi)^2}2\pi\int_0^\infty kdk\frac{\delta(k-\sqrt{2E/m})}{|\hbar^2k/m|} \tag{17}$$

$$= A\frac{m}{2\pi\hbar^2} \tag{18}$$

Thus we see that in 2D the density of states is constant for $\mathcal{E} > 0$ and zero for $\mathcal{E} < 0$.

(b) Expressing the density of bosons as a sum over Bose occupation factors we have

$$\frac{N}{A} = \frac{1}{A}\sum_k n(\mathcal{E}(k)) = \frac{1}{A}n(0) + \int\frac{d^2k}{(2\pi)^2}\frac{1}{z_1e^{\beta\hbar^2k^2/2m} - 1} \tag{19}$$

where, in the second equality, we have separated out the occupation of the ground state $n(0)$ and converted the remaining sum over $\vec{k}$ to an integral.

Note that

$$n(0) = \frac{1}{z^{-1} - 1} = \frac{z}{1-z} \tag{20}$$

If we substitute the above in for $n(0)$ and, using the density of states from (a) convert the integral over k into an integral over $\mathcal{E}$, we have

$$\frac{N}{A} = \frac{z}{1-z}\frac{1}{A} + \frac{2\pi m}{h^2}\int_0^\infty d\mathcal{E}\frac{1}{z^{-1}e^{\beta\mathcal{E}} - 1} \tag{21}$$

Finally, making the change of variables $y = \beta\mathcal{E} = \mathcal{E}/(k_BT)$ in the integral, we obtain the desired result

$$\frac{N}{A} = \frac{z}{1-z}\frac{1}{A} + \frac{2\pi k_BTm}{h^2}\int_0^\infty dy\frac{1}{z^{-1}e^y - 1} \tag{22}$$

(c) The integral appearing in (22) can be done analytically with the result

$$\int_0^\infty dy\frac{1}{z^{-1}e^y - 1} = \ln\frac{1}{1-z} \tag{23}$$

Thus we have

$$\frac{N}{A} = \frac{N_0}{A} - \frac{1}{\lambda^2}\ln(1-z) \tag{24}$$

In the thermodynamic limit, at $T = T_c$ the condensate density N_0/A vanishes, and the chemical potential μ goes to zero so the fugacity z goes to 1. Because the integral (23) diverges as $z \to 1$, it follows that $T_c = 0$ for this gas.

(d) Because this 2D gas does not exhibit Bose condensation, at any finite temperature in the thermodynamic limit the condensate density vanishes. Thus we have

$$\frac{N}{A} = \frac{1}{l^2} = \frac{1}{\lambda^2}\ln\frac{1}{1-z} \tag{25}$$

Here $l = (A/N)^{1/2}$ is the mean interparticle spacing, and $\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$ is the thermal wavelength. Solving for the fugacity z we obtain

$$z = 1 - e^{-\lambda^2/l^2} \tag{26}$$

In the classical limit, i.e. the high temperature / low density limit, $\lambda^2/l^2 \ll 1$. Thus in this limit

$$z \simeq 1 - \left(1 - \frac{\lambda^2}{l^2}\right) = \frac{\lambda^2}{l^2} \ll 1 \tag{27}$$

as expected.

Problem 3

(a) As in the previous problem, we use the fact that in the thermodynamic limit, at T_c the fugacity is 1 ($z = 1$) and the condensate density vanishes ($N_0/A = 0$) to write down the following equation which we can solve to find T_c

$$\frac{N}{A} = \int \frac{d^2k}{(2\pi)^2}\frac{1}{e^{\beta c(\hbar k)^{3/2}} - 1} \tag{28}$$

$$= \int \frac{d^2k}{(2\pi)^2}\frac{1}{e^{\alpha k^{3/2}/T} - 1} \tag{29}$$

where $\alpha = c\hbar^{3/2}/k_B$.

Making the change of variables $\vec{q} = \vec{k}/T^{2/3}$ we find

$$\frac{N}{A} = T_c^{4/3}\int \frac{d^2q}{(2\pi)^2}\frac{1}{e^{\alpha q^{3/2}} - 1} \tag{30}$$

For T_c to be finite it is only necessary that the integral over q appearing in the above expression be finite. The only "dangerous" limit of the integrand is for small q. In this limit we can Taylor expand $e^{\alpha q^{3/2}}$ to find that

$$\int \frac{d^2q}{(2\pi)^2}\frac{1}{e^{\alpha q^{3/2}} - 1} \quad \int_0 q dq \frac{1}{q^{3/2}} \quad \int_0 \frac{dq}{q^{1/2}} = \text{finite.} \tag{31}$$

The point is that the small q singularity in the integrand goes as $1/q^{1/2}$, which is integral. (This is in contrast to the previous problem, for which the same analysis would give a singularity of the form $1/q$ which is *not* integrable.) Because this integral is finite we conclude that T_c is finite.

Given that T_c is finite, we can also conclude from the above that

$$T_c \propto n^{3/4} \tag{32}$$

where $n = N/A$ is the number density of bosons.

(b) The grand partition function for bosons with dispersion $\mathcal{E}(\vec{k})$ is

$$\mathcal{Z} = \prod_{\vec{k}} \frac{1}{1 - ze^{-\beta\mathcal{E}(k)}} \tag{33}$$

The grand potential is then

$$\Sigma = -k_B T \ln \mathcal{Z} = k_B T \sum_{\vec{k}} \ln(1 - z e^{-\beta \mathcal{E}(k)}) \tag{34}$$

Separating out the contribution of the condensate and turning the sum into an integral in the usual way we obtain

$$\Sigma = -k_B T \ln(1 - z) - k_B T A \int \frac{d^2k}{(2\pi)^2} \ln(1 - z e^{-\beta \mathcal{E}(k)}) \tag{35}$$

Note that

$$N_0 + 1 = \frac{z}{1 - z} + 1 = \frac{1}{1 - z} \tag{36}$$

where $N_0 = n(0)$ is the occupation number of the ground state. It follows that

$$\Sigma = k_B T \ln(N_0 + 1) - k_B T A \int \frac{d^2k}{(2\pi)^2} \ln(1 - z e^{-\beta \mathcal{E}(k)}) \tag{37}$$

The first term is $\propto \ln N_0$, while the second term is $\propto A$, where A is the area of the system. In the thermodynamic limit the first term, which is not extensive, is negligible when compared with the second term which *is* extensive. For $T < T_C$, again in the thermodynamic limit, the fugacity z is equal to 1. Thus, for $T < T_C$, in the thermodynamic limit the grand potential is given by

$$\Sigma \;=\; -k_B T \int \frac{d^2k}{(2\pi)^2} \ln(1 - e^{-\alpha k^{3/2}/T}) \tag{38}$$

$$\tag{39}$$

where α is as defined in (a) above.

To analyze Σ without actually doing the integral, we can again make the change of variables $\vec{q} = \vec{k}/T^{2/3}$, with the result

$$\Sigma = -(k_B T) T^{4/3} A \int \frac{d^2q}{(2\pi)^2} \ln(1 - e^{-\alpha q^{3/2}}) \propto A T^{7/3} \tag{40}$$

Thus we see that for $T < T_C$, $\Sigma \propto T^{7/3}$. It follows that the entropy scales as

$$S = -\left(\frac{\partial \Sigma}{\partial T}\right)_{\mu, A} \propto T^{4/3} \tag{41}$$

and the pressure scales as

$$P = -\frac{\Sigma}{A} \propto T^{7/3} \tag{42}$$