PHY5524
Problem Set 9: Solution

Problem 1

(a) The density of states in D dimensions for particles with energy dispersion $\mathcal{E} = A|\vec{k}|^s$ is

$$a(\mathcal{E}) \;=\; L^D \int \frac{d^D k}{(2\pi)^D}\delta(\mathcal{E} - A|\vec{k}|^s) \tag{1}$$

where the "volume" V is equal to L^D (where L is the linear dimension of the system). Using the identity $\delta(Cx) = \frac{1}{C}\delta(x)$ we then have

$$a(\mathcal{E}) \;=\; \mathcal{E}^{-1} V \int \frac{d^D k}{(2\pi)^D}\delta(1 - A|\vec{k}|^s/\mathcal{E}). \tag{2}$$

Next, making the change of variables $\vec{q} = (A/\mathcal{E})^{1/s}\vec{k}$ we obtain

$$a(\mathcal{E}) \;=\; \mathcal{E}^{D/s-1}\frac{1}{A^{1/s}}\int \frac{d^D q}{(2\pi)^D}\delta(1 - |\vec{q}|^s) \tag{3}$$

Since the integral no longer depends on $\mathcal{E}$ we see that

$$a(\mathcal{E}) \propto \mathcal{E}^{\frac{D}{s}-1} \tag{4}$$

(b) At T_c for Bose condensation the chemical potential equals zero, $\mu = 0$, and the condensate density N_0/V vanishes. Thus, the condition for T_c is that

$$\frac{N}{V} = \int \frac{d^D k}{(2\pi)^D}\frac{1}{e^{\beta_c \mathcal{E}(\vec{k})} - 1} \tag{5}$$

where $\beta_c = 1/k_B T_c$. Using the result for the density of states obtained in Part (a) and making the change of variables $y = \beta\mathcal{E}$ we obtain,

$$\frac{N}{V} \;=\; C\int_0^\infty \mathcal{E}^{D/s-1}\frac{1}{e^{\beta_c E} - 1}d\mathcal{E} = C T_c^{D/s}\int_0^\infty \frac{y^{D/s-1}dy}{e^y - 1}. \tag{6}$$

For T_c to be finite it is only necessary that the integral over y in the second equality above must converge. The "dangerous" part of the integral is when $y \to 0$ so we can determine if the integral is finite or not by considering its behavior in this limit. Taylor expanding the e^y in the denominator we see that the behavior of the integral at the lower limit of integration is

$$\int_0 \frac{y^{D/s-1}}{(1 + y + \cdots) - 1}dy \sim \int_0 y^{D/s-2}dy \sim y^{D/s-1}\Big|_{y=0} \tag{7}$$

This will be finite provided

$$\frac{D}{s} - 1 > 0 \tag{8}$$

or, equivalently, $D > s$.

(c) For $T < T_c$ the chemical potential remains zero. From Part (b) we see that the number density of excited bosons satisfies

$$\frac{N_e}{V} = \frac{N}{V}\left(\frac{T}{T_C}\right)^{D/s} \tag{9}$$

Thus we have

$$\frac{N}{V} = \frac{N_0}{V} + \frac{N}{V}\left(\frac{T}{T_C}\right)^{D/s} \tag{10}$$

from which we find the condensate fraction for $T < T_c$ is given by

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{D/s} \tag{11}$$

(d) The grand partition function for an ideal Bose gas is

$$\mathcal{Z} = \prod_{\vec{k}} \ln \frac{1}{1 - ze^{-\beta\mathcal{E}(\vec{k})}} \tag{12}$$

The grand potential is then given by

$$\Sigma = -k_B T \ln \mathcal{Z} = k_B T \ln(1 - z) + k_B T V \int \frac{d^D k}{(2\pi)^D} \ln(1 - ze^{-\beta\mathcal{E}(\vec{k})}) \tag{13}$$

where $V = L^D$ is the "volume" of the D-dimensional gas. Using the fact that $\Sigma = -PV$ as well as the fact that $1 - z = 1/(1 + N_0)$ we find

$$P \;=\; -\frac{\Sigma}{V} = k_B T \frac{\ln(1 + N_0)}{V} - k_B T \int \frac{d^D k}{(2\pi)^D} \ln(1 - ze^{-\beta A|\vec{k}|^s}) \tag{14}$$

The first term (the contribution to the pressure due to the condensate) goes to zero in the thermodynamic limit. Thus we have, after turning the k sum into an energy integration using the density of states $a(\mathcal{E}) = A\mathcal{E}^{D/s-1}$,

$$P \;=\; -k_B T A \int_0^\infty \mathcal{E}^{D/s-1} d\mathcal{E} \ln(1 - e^{-\beta\mathcal{E}}) \tag{15}$$

Letting $y = \beta\mathcal{E}$ then gives

$$P \;=\; -k_B T (k_B T)^{D/s} C \int_0^\infty y^{D/s-1} dy \ln(1 - e^{-y}) \tag{16}$$

The integral no longer depends on T and so is just a number. Thus we see that

$$P \propto T^{D/s+1} \tag{17}$$

Problem 2

(a) At $T = T_c$ the condensate density is zero and $\mu = 0$. Thus we can find T_c by solving the equation

$$\frac{N}{V} \;=\; \sum_{\vec{k}} \frac{1}{e^{\beta\mathcal{E}(\vec{k})} - 1} + \sum_{\vec{k}} \frac{1}{e^{\beta(\mathcal{E}(\vec{k})+\Delta)} - 1} \tag{18}$$

Here $\mathcal{E}(\vec{k}) = \hbar^2 k^2/2m$. The first term is the number density of bosons for which the internal degree of freedom is in its ground state, and the second term is the number density of bosons for which the internal degree of freedom is in its excited state.

Turing the sums into integrals and using the fact that $\Delta \gg k_B T_c$ we then have

$$\frac{N}{V} \;\simeq\; \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\beta\mathcal{E}(\vec{k})} - 1} + e^{-\beta\Delta} \int \frac{d^3 k}{(2\pi)^3} e^{-\beta\mathcal{E}(\vec{k})} \tag{19}$$

$$= \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \zeta(3/2) + e^{-\Delta/k_B T} \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \tag{20}$$

This equation is satisfied when $T = T_c$. Thus we have

$$T_c \;=\; \left(\frac{N}{V}\right)^{2/3} \frac{h^2}{2\pi m k_B} \left(\frac{1}{\zeta(3/2) + e^{-\Delta/k_B T_c}}\right)^{2/3} \tag{21}$$

$$\simeq \left(\frac{N}{V}\frac{1}{\zeta(3/2)}\right)^{2/3} \frac{h^2}{2\pi m k_B} \left(1 - \frac{2}{3\zeta(3/2)} e^{-\Delta/k_B T_c}\right) \tag{22}$$

where in the second equality we have Taylor expanded in powers of $e^{-\Delta/k_BT}$ since we are assuming $\Delta \gg k_BT$. Finally, in this limit, we can, to an excellent approximation, simply replace T_c on the right hand side of the above equation by its value when $\Delta \to \infty$. Thus we have

$$T_c \simeq \left(\frac{N}{V}\frac{1}{\zeta(3/2)}\right)^{2/3} \frac{h^2}{2\pi m k_B}\left(1 - \frac{2}{3\zeta(3/2)}e^{-\Delta/k_BT_c^0}\right) \tag{23}$$

where

$$T_c^0 = \left(\frac{N}{V}\frac{1}{\zeta(3/2)}\right)^{2/3} \frac{h^2}{2\pi m k_B} \tag{24}$$

is the Bose condensation temperature of a single component Bose gas.

(b) From Part (a) we find that the change in T_c due to the internal degree of freedom is

$$T_c - T_c^0 = -T_c^0 \frac{2}{3\zeta(3/2)}e^{-\Delta/k_BT_c^0} \tag{25}$$

Thus T_c *decreases* due to internal degree of freedom.

Problem 3.

In class we showed that for adiabatic processes (S = Const.) the ideal Bose gas obeys the law $Pv^{5/3}$ = Const., where $v = V/N$ is the volume per particle. Since the mass density is $\rho = mN/V = m/v$ this implies that for adiabatic processes

$$P = C\rho^{5/3} \tag{26}$$

It follows that

$$\left(\frac{\partial P}{\partial \rho}\right)_S = \frac{5}{3}C\rho^{2/3} = \frac{5}{3}\frac{P}{\rho} \tag{27}$$

For $T > T_c$ it was shown in class that

$$\frac{N}{V} = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} g_{3/2}(z) \tag{28}$$

and

$$P = k_BT\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} g_{5/2}(z) \tag{29}$$

Since $\rho = mN/V$ it follows that

$$\frac{P}{\rho} = \frac{1}{m}\frac{P}{N/V} = \frac{1}{m}\frac{k_BT g_{5/2}(z)}{g_{3/2}(z)} \tag{30}$$

Thus, if w is the speed of sound in the gas, we find that

$$w^2 = \left(\frac{\partial P}{\partial \rho}\right)_S = \frac{5k_BT}{3m}\frac{g_{5/2}(z)}{g_{3/2}(z)} \tag{31}$$

The mean square velocity of a particle in an ideal Bose gas is

$$< u^2 > = \frac{1}{N}\sum_{\vec{k}}(\hbar\vec{k}/m)^2 \frac{1}{z^{-1}e^{\beta\mathcal{E}(\vec{k})} - 1} = \frac{1}{N}\int_0^\infty d\mathcal{E}a(\mathcal{E})\frac{2\mathcal{E}}{m}\frac{1}{z^{-1}e^{\beta\mathcal{E}} - 1} \tag{32}$$

Since $a(\mathcal{E}) = C\mathcal{E}^{1/2}$ and making the change of variables $y = \beta\mathcal{E}$ we find

$$< u^2 > = \frac{1}{N}\int_0^\infty d\mathcal{E}C\mathcal{E}^{1/2}\frac{2\mathcal{E}}{m}\frac{1}{z^{-1}e^{\beta\mathcal{E}} - 1} = \frac{1}{N}C\frac{2(k_BT)^{5/2}}{m}\int_0^\infty \frac{y^{3/2}}{z^{-1}e^y + 1}dy \tag{33}$$

Likewise, we have that the number of bosons is given by

$$N = \sum_{\vec{k}} \frac{1}{z^{-1}e^{\beta\mathcal{E}(\vec{k})} - 1} = \int_0^\infty d\mathcal{E} a(\mathcal{E}) \frac{1}{z^{-1}e^{\beta\mathcal{E}} - 1} \tag{34}$$

Again using the fact that $a(\mathcal{E}) = C\mathcal{E}^{1/2}$ and making the change of variables $y = \beta E$ we find

$$N = C(k_B T)^{3/2} \int_0^\infty \frac{y^{1/2}}{z^{-1}e^y - 1} \tag{35}$$

Finally, using the fact that, by definition

$$g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{y^{1/2}}{z^{-1}e^y - 1} dy \tag{36}$$

and

$$g_{5/2}(z) = \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{y^{3/2}}{z^{-1}e^y - 1} dy \tag{37}$$

we find

$$< u^2 >= \frac{3k_B T}{m} \frac{g_{5/2}(z)}{g_{3/2}(z)} \tag{38}$$

from which it follows that

$$w^2 = \frac{5}{9} < u^2 > . \tag{39}$$