**Statistical Mechanics** — PHY 5524

Final Exam April 27, 2009

**1.** (50 pts) Consider a three-dimensional gas of $N$ noninteracting bosons in a volume $V$ with energy dispersion $\mathcal{E}(\mathbf{k}) = \hbar v|\mathbf{k}|$ where $v$ has units of velocity.

(a) Determine the one-particle density of states for this gas.

(b) Show that this gas exhibits Bose condensation at a finite temperature $T_c \propto (N/V)^{\gamma}$ and determine the exponent $\gamma$.

(c) For $T < T_c$ obtain an expression for the pressure, $P$, of this gas. Use the fact that $\Sigma = -PV$ where $\Sigma$ is the grand potential, $\Sigma = -k_B T \ln \mathcal{Z}$, and $\mathcal{Z}$ is the grand partition function. Express your answer in terms of an integral over energy $\mathcal{E}$. (Do not try to perform this integral).

(d) For $T < T_c$ find an expression for the total energy, $E$, of this gas, expressed as an integral over energy $\mathcal{E}$ (Again, do not try to perform this integral).

(e) For $T < T_c$, using the results of Parts (c) & (d), show that $PV = \alpha E$ and determine the value of $\alpha$. (**Hint:** Perform an integration by parts.)

**2.** (50 pts) Consider a two-dimensional gas of $N$ noninteracting spin-1/2 fermions in an area $A$ with non-relativistic energy dispersion $\mathcal{E}(\mathbf{k}) = \hbar^2|\mathbf{k}|^2/(2m)$.

(a) Find an expression for the Fermi energy, $\mathcal{E}_F$, of this gas expressed as a function of $n = N/A$, $m$, and fundamental constants.

(b) Obtain an exact expression for the chemical potential, $\mu$, of this gas valid at arbitrary temperature.

(c) Show that in the limit $T \to 0$ the expression for $\mu$ obtained in Part (b) becomes equal to the Fermi energy obtained in Part (a).

(d) For $k_BT \ll \mathcal{E}_F$ the specific heat of this gas is

$$C_V = \frac{\pi^2}{3} N k_B \left(\frac{k_BT}{\mathcal{E}_F}\right),$$

(you do not need to show this). In one or two sentences explain qualitatively why this specific heat is so much lower than the specific heat of a classical 2D monatomic gas ($C_V^{\text{cl.}} = Nk_B$).

**Hint**: For Part (b) you may find the following integral useful:

$$\int_0^\infty \frac{1}{z^{-1}e^y + 1} dy = \ln(1+z).$$