Statistical Mechanics — PHY 5524

2008 Final Exam Solution

1. (a) The grand partition function for a gas of photons is

$$\mathcal{Z} = \prod_{\mathbf{k},s} \frac{1}{1 - e^{-\beta\hbar c|\vec{k}|}} \tag{1}$$

and so the grand potential is

$$\Sigma = -k_B T \ln \mathcal{Z} = k_B T \sum_{\mathbf{k},s} \ln(1 - e^{-\beta\hbar ck}) = k_B T \int_0^\infty d\omega a(\omega) \ln(1 - e^{-\beta\hbar\omega}) \tag{2}$$

where $a(\omega) = V\omega^2/(\pi^2 c^3)$ (as given in the problem).

(b) Making the change of variables $y = \beta\hbar\omega$ we find that

$$\Sigma = k_B T \frac{V}{\pi^2 c^3} \left(\frac{k_B T}{\hbar}\right)^3 \int_0^\infty dy y^2 \ln(1 - e^{-y}) \tag{3}$$

from which we see that $\Sigma \propto VT^4$ and so $\alpha = 4$.

(c) Since $S = -(\partial\Sigma/\partial T)_{\mu,V}$ it follows that $S \propto VT^3$ and so $\beta = 3$.

(d) For adiabatic processes we see from Part (c) that $VT^3 = \text{Const}$. It follows that

$$V_f = V_i \frac{T_i^3}{T_f^3} = 10^9 V_i \tag{4}$$

2. (a) If the Fermi wave vector is k_F than the total number of particles is

$$N = \frac{L}{(2\pi)} 2k_F \tag{5}$$

It follows that $k_F = \pi n$ where $n = N/L$. The Fermi energy is then

$$\mathcal{E}_F = \frac{\hbar^2 \pi^2 n^2}{2m} \tag{6}$$

(b) We find the density of states as follows

$$a(\mathcal{E}) \;=\; 2L \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \delta(\mathcal{E} - \hbar^2 k^2/(2m)) = \frac{2L}{\pi} \frac{1}{(\hbar^2/2m)(\sqrt{2m\mathcal{E}/\hbar^2})} = \frac{2L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{\sqrt{\mathcal{E}}} \tag{7}$$

(c) Since the density of states has the form $a(\mathcal{E}) = C\mathcal{E}^{-1/2}$ it follows that

$$N = \int_0^{\mathcal{E}_F} a(\mathcal{E}) d\mathcal{E} = C \; 2\mathcal{E_F}^{1/2} \tag{8}$$

and

$$E = \int_0^{\mathcal{E_F}} a(\mathcal{E}) \mathcal{E} d\mathcal{E} = C \; \frac{2}{3} \mathcal{E_F}^{3/2} \tag{9}$$

Thus we have

$$E = \frac{1}{3} N \mathcal{E}_F \tag{10}$$

and so $q = 1/3$.

(d) Using the results of Parts (a) & (c) we find that $E \propto N^3/L^2$. It follows that at $T = 0$ the pressure is $P = -(\partial E/\partial L) \propto n^3$, and so $\alpha = 3$.

3. At T_c, we know that in the thermodynamic limit $z = 1$ and $N_0/V = 0$. Thus for the given density of states we have

$$n = \frac{N}{V} = g \int_0^\infty d\mathcal{E} \frac{\mathcal{E}^\nu}{e^{\beta \mathcal{E}} - 1} \tag{11}$$

Letting $y = \beta \mathcal{E}$ we find

$$n = g(k_B T_c)^{\nu+1} \int_0^\infty dy \frac{y^\nu}{e^y - 1} \tag{12}$$

If $\nu > 0$ the integral over y converges and is just a dimensionless number. It follows that $T_c \propto n^{1/(\nu+1)}$