

Physics 5524
Statistical Mechanics
Problem Set 11
Due: Monday, Apr. 13

11.1 Consider a gas of photons in thermal equilibrium with the walls of a cavity of volume V at temperature T .

- (a) Write down the grand partition function $\mathcal{Z}(V, T)$ for this gas. (Remember that for a gas of photons $\mu = 0$ and so $z = 1$).
- (b) Find an expression for the grand potential Σ for this gas. Express your result as a function of T , V and an integral over frequency ω . (You need not perform the integral).
- (c) Find an expression for the total energy E of this gas, again expressed as a function of T , V and an integral over frequency ω .
- (d) Using the results of Parts (b) & (c) show that $PV = \frac{1}{3}E$ for this gas. (**Hint:** To do this you will need to integrate by parts.)

11.2 (From the Spring 2004 Comprehensive Exam)

Consider a degenerate, ultra-relativistic ($mc^2 \ll cp_F$) gas of non-interacting electrons. In this limit, the energy of an electron is related to its momentum by $\mathcal{E}(\vec{p}) = c|\vec{p}|$. Consider N such electrons in a volume V .

- (a) At zero temperature, Find the chemical potential μ and the Fermi momentum p_F for this gas as a function of N and V .
- (b) Find the total energy of the system E at zero temperature as a function of N and V .
- (c) Use the fact that at $T = 0$ the pressure is given $P = -\frac{\partial E}{\partial V}$ to obtain an expression for P and show that $PV = \frac{1}{3}E$ when $T = 0$.

- (d) Write down the grand partition function for this gas at finite temperature, and use it to obtain an expression for the grand potential Σ . Express Σ as a function of T , V , μ and an integral over energy. (Do not try to do the integral.)
- (e) Find an expression for the total energy E for this gas. Again, your answer should be expressed as a function of T , V , μ and an integral over energy. (Again, do not try to do the integral.)
- (f) Using the results of Parts (d) and (e) show that $PV = \frac{1}{3}E$ for any temperature T (not just $T = 0$). As in Prob. 11.1, to do this you will have to perform an integration by parts.

Note: Problems 11.1 and 11.2 illustrate a generic feature of ideal gases which holds regardless of whether they are described by Bose-Einstein, Fermi-Dirac or Maxwell-Boltzmann statistics. For an ideal gas in D dimensions, if the energy dispersion of the particles is $\mathcal{E}(\vec{k}) = A|\vec{k}|^s$ then $PV = \frac{s}{D}E$.

11.3 Consider an ideal gas of N spin-1/2 fermions of mass m in *two* space dimensions. Let A be the area of the gas so that the particle density is $n = N/A$.

- (a) Determine the Fermi wave vector k_F , the Fermi energy E_F and the Fermi temperature T_F for this gas.
- (b) Determine the density of states $a(\mathcal{E})$ for this gas.
- (c) Obtain a closed form expression for the chemical potential μ for this gas as a function of temperature. Show that in the limit of high temperature ($T \gg T_F$) the fugacity is approximately given by $z \simeq \lambda^2/l^2$ where λ is the thermal wave length and $l = \sqrt{A/N}$ is the mean particle spacing.
- (d) Obtain the specific heat at constant area of this gas in both the low temperature $T \ll T_F$ and high temperature $T \gg T_F$ limits.