PHY5524
Problem Set 10: Solution

Problem 1

(a) The number of bosons is given by the usual sum over Bose occupation factors,

$$N = \sum_{n_x,n_y,n_z} \frac{1}{z^{-1}e^{\beta\hbar\omega_0(n_x+n_y+n_z)} - 1} \tag{1}$$

If, in this sum, we let $x_{n_x} = n_x\beta\hbar\omega_0$, $y_{n_y} = n_y\beta\hbar\omega_0$ and $z_{n_z} = n_z\beta\hbar\omega_0$ and $\Delta x = x_{n_x+1} - x_{n_x} = \beta\hbar\omega_0$, $\Delta y = y_{n_y+1} - y_{n_y} = \beta\hbar\omega_0$ and $\Delta z = z_{n_z+1} - z_{n_z} = \beta\hbar\omega_0$ then Eq. (1) can be written

$$N = \left(\frac{k_BT}{\hbar\omega_0}\right)^3 \sum_{n_x=0}^{\infty} \Delta x \sum_{n_y=0}^{\infty} \Delta y \sum_{n_z=0}^{\infty} \Delta z \frac{1}{z^{-1}e^{x_{n_x}+y_{n_y}+z_{n_z}} - 1} \tag{2}$$

In the limit $k_BT \gg \hbar\omega_0$, $\Delta x, \Delta y, \Delta z \to 0$ and the sums becomes integrals with the result

$$N = N_0 + \left(\frac{k_BT}{\hbar\omega_0}\right)^3 \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{1}{z^{-1}e^{x+y+z} - 1} \tag{3}$$

where we have separated out the occupation of the ground state ($n_x = n_y = n_z = 0$),

$$N_0 = \frac{z}{1-z} \tag{4}$$

(b) At T_c the condensate fraction N_0/N vanishes and $z = 1$ in the thermodynamic ($N \gg 1$) limit. Thus we can determine T_C by solving the following equation

$$N = \left(\frac{k_BT_c}{\hbar\omega_0}\right)^3 \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{1}{z^{-1}e^{x+y+z} - 1} = \left(\frac{k_BT_c}{\hbar\omega_0}\right)^3 \zeta(3) \tag{5}$$

from which we find

$$T_C = \frac{\hbar\omega_0}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3} \tag{6}$$

If $N \gg 1$ then

$$k_BT_C = \hbar\omega_0\left(\frac{N}{\zeta(3)}\right)^{1/3} \gg \hbar\omega_0 \tag{7}$$

and so we see that the assumption that $k_BT_c \gg \hbar\omega_0$ used to turn the sum into an integral was, in fact, justified.

(c) For $T < T_c$ we still have $z = 1$ but there is now a macroscopic occupation of the ground state. Thus we have

$$N = N_0 + \left(\frac{k_BT}{\hbar\omega_0}\right)^3 \zeta(3) = N_0 + N\left(\frac{T}{T_C}\right)^3 \tag{8}$$

where, in the last equality, we have used the result from Part (b) for T_c. Solving for the condensate fraction we find that, for $T < T_c$,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_C}\right)^3 \tag{9}$$

(d) For $T < T_c$ the fugacity is 1 ($z = 1$) and so the total energy is given by

$$\begin{aligned} E &= \sum_{n_x,n_y,n_z} \mathcal{E}_{n_x,n_y,n_z} n(\mathcal{E}_{n_x,n_y,n_z}) && (10) \\ &= \sum_{n_x,n_y,n_z} \frac{(n_x+n_y+n_z)\hbar\omega_0}{e^{\beta\hbar\omega_0(n_x+n_y+n_z)} - 1} && (11) \end{aligned}$$

Again we can turn the sum into an integral. Note that here there is no need to separate out the contribution of the ground state, since the ground state has zero energy. Following the same procedure as in Part (a), we therefore find that

$$E = k_BT\left(\frac{k_BT}{\hbar\omega_0}\right)^3\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\frac{x+y+z}{e^{x+y+z}-1} = \frac{(k_BT)^4}{(\hbar\omega_0)^3}3\zeta(4) \tag{12}$$

The specific heat is then

$$C = \frac{\partial E}{\partial T} = 12\zeta(4)\frac{k_B^4}{(\hbar\omega_0)^3}T^3 \tag{13}$$

Note that we could also have expressed the total energy as

$$E = \frac{3\zeta(4)}{\zeta(3)}Nk_BT\left(\frac{T}{T_C}\right)^3 \tag{14}$$

and the specific heat as

$$C = \frac{\partial E}{\partial T} = \frac{12\zeta(4)}{\zeta(3)}Nk_B\left(\frac{T}{T_C}\right)^3 \tag{15}$$

(e) For a trap with $\omega_0 = 2\pi \times 100\mathrm{s}^{-1}$ and $N = 10^6$ the condensation temperature is

$$T_c = \frac{\hbar\omega_0}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3} = \frac{(1.05\times10^{-34}J-s)2\pi(100s^{-1})}{1.4\times10^{-23}J/K}\left(\frac{10^6}{1.202}\right)^{1/3} = 4.4\times10^{-7}K \tag{16}$$

Problem 2.

(a) For a two dimensional solid consisting of N ions we expect $2N$ normal modes. We can then find the Debye wavevector k_D from the following

$$2N = \sum_{\vec{k},s} = 2A\int\frac{d^2k}{(2\pi)^2} = 2A\frac{1}{(2\pi)^2}2\pi\int_0^{k_D}kdk = 2A\frac{1}{(2\pi)^2}\pi k_D^2 \tag{17}$$

Thus we have

$$k_D = \left(4\pi\frac{N}{A}\right)^{1/2} \tag{18}$$

The Debye frequency is then

$$\omega_D = ck_D = c\left(4\pi\frac{N}{A}\right)^{1/2} \tag{19}$$

and the Debye temperature is

$$\Theta_D = \hbar\omega_D/k_B = \frac{\hbar c}{k_B}\left(4\pi\frac{N}{A}\right)^{1/2} \tag{20}$$

(b) The total energy of the phonons in this solid is given by (here we ignore the zero point energy of the oscillators, which just gives a temperature independent constant),

$$E \;=\; 2\sum_{\vec{k}}\frac{\hbar\omega_s(\vec{k})}{e^{\beta\hbar\omega_s(\vec{k})}-1} = 2A\int\frac{d^2k}{(2\pi)^2}\frac{\hbar c|\vec{k}|}{e^{\beta\hbar c|\vec{k}|}-1} = 2A\frac{1}{(2\pi)^2}2\pi\int_0^{k_D}kdk\frac{\hbar ck}{e^{\beta\hbar ck}-1} \tag{21}$$

The specific heat is then

$$C_V = \frac{\partial E}{\partial T} = -\frac{1}{k_BT^2}\frac{\partial E}{\partial\beta} = N\frac{2}{k_D^2}\frac{1}{k_BT^2}\int_0^{k_D}kdk\frac{(\hbar ck)^2e^{\beta\hbar ck}}{(e^{\beta\hbar ck}-1)^2} \tag{22}$$

where we have used the fact that $A = 4\pi N/k_D^2$. Next, making the change of variables $x = \beta\hbar c k = \hbar c k/(k_B T)$ we find

$$C_V = 4Nk_B \frac{1}{k_D^2}\frac{1}{(k_B T)^2}(\hbar c)^2 \left(\frac{k_B T}{\hbar c}\right)^4 \int_0^{\Theta_D/T} \frac{x^3 e^x}{(e^x-1)^2} dx \tag{23}$$

which can be simplified to obtain

$$C_V = 4Nk_B \left(\frac{T}{\Theta_D}\right)^2 \int_0^{\Theta_D/T} \frac{x^3 e^x}{(e^x-1)^2} dx \tag{24}$$

For $T \ll \Theta_D$

$$\int_0^{\Theta_D/T} \frac{x^3 e^x}{(e^x-1)^2} dx \simeq \int_0^{\infty} \frac{x^3 e^x}{(e^x-1)^2} dx = 6\zeta(3) \tag{25}$$

and we find

$$C_V \simeq 24\zeta(3) N k_B \left(\frac{T}{\Theta_D}\right)^2 \tag{26}$$

For $T \gg \Theta_D$

$$\int_0^{\Theta_D/T} \frac{x^3 e^x}{(e^x-1)^2} dx \simeq \int_0^{\Theta_D/T} \frac{x^3(1+\cdots)}{((1+x\cdots)-1)^2} dx = \int_0^{\Theta_D/T} x dx = \frac{1}{2}\left(\frac{\Theta_D}{T}\right)^2 \tag{27}$$

and we find the expected classical equipartition result (a 2D version of the law of Dulong and Petit),

$$C_V \simeq 2Nk_B. \tag{28}$$

Problem 3

For phonons with dispersion $\omega(\vec{k}) = c|\vec{k}|^s$ the density of states (defined so that $g(\omega)d\omega$ is the number of normal modes with angular frequencies between ω and $\omega + d\omega$) is

$$g(\omega) = 3V \int \frac{d^3k}{(2\pi)^3} \delta(\omega - c|\vec{k}|^s) = 3\frac{V}{(2\pi)^3} 4\pi \int_0^\infty k^2 dk \frac{\delta(k-(\omega/c)^{1/s})}{sck^{s-1}} = D\omega^{\frac{3-s}{s}} \tag{29}$$

where D is a constant.

The total energy of a Debye solid with these phonons is (again ignoring the zero point energy of each oscillator, which just contributes a T-independent constant),

$$E = \int_0^{\omega_D} d\omega g(\omega) \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \tag{30}$$

The specific heat is then

$$C_V = \frac{\partial E}{\partial T} = \frac{1}{k_B T^2} \int_0^{\omega_D} d\omega g(\omega) \frac{(\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} \tag{31}$$

Inserting the above expression for the density of states we then find,

$$C_V = D\frac{1}{k_B T^2} \int_0^{\omega_D} \omega^{\frac{3-s}{s}} \frac{(\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} d\omega \tag{32}$$

Next, making the change of variables $y = \beta\hbar\omega = \hbar\omega/(k_B T)$ yields

$$C_V = D\hbar^2 \frac{1}{k_B T^2} \left(\frac{k_B T}{\hbar}\right)^{\frac{3}{s}+2} \int_0^{\theta_D/T} \frac{x^{\frac{3+s}{s}} e^x}{(e^x-1)^2} dx \tag{33}$$

At low temperatures ($T \ll \theta_D$) the upper limit of the integral can be taken to infinity. The result is just a dimensionless number. We therefore find that, at low temperatures, the specific heat obeys the law

$$C_V = \alpha T^{\frac{3}{s}} \tag{34}$$