PHY5524
Problem Set 11: Solution

Problem 1

(a) The grand partition function for a gas of photons is

$$\mathcal{Z}(V,T) = \prod_{\vec{k},s} \frac{1}{1 - e^{-\beta\hbar\omega(\vec{k})}} \tag{1}$$

where $\omega(\vec{k}) = c|\vec{k}|$ and s labels the two polarization states of each photon. (Note that the chemical potential for photons is 0.)

(b) The grand potential for this gas is then

$$\Sigma = -k_BT\ln\mathcal{Z} = k_BT\sum_{\vec{k},s}\ln(1-e^{-\beta\hbar c|\vec{k}|}) = k_BT\int d\omega g(\omega)\ln(1-e^{-\beta\hbar\omega}) \tag{2}$$

where

$$g(\omega) = \frac{V}{\pi^2}\frac{\omega^2}{c^3} \tag{3}$$

is the photon density of states (defined so that $g(\omega)d\omega$ is equal to the number of states with angular frequency between ω and $\omega + d\omega$). Thus we have

$$\Sigma = k_BT\frac{V}{\pi^2}\frac{1}{c^3}\int_0^\infty d\omega\omega^2\ln(1-e^{-\beta\hbar\omega}) \tag{4}$$

(c) The total energy of this gas is given by

$$E = \sum_{\vec{k},s}\hbar\omega(\vec{k})\frac{1}{e^{\beta\hbar\omega(\vec{k})}-1} = \int_0^\infty d\omega g(\omega)\frac{\hbar\omega}{e^{\beta\hbar\omega}-1} = \frac{V}{\pi^2}\frac{\hbar}{c^3}\int_0^\infty d\omega\frac{\omega^3}{e^{\beta\hbar\omega}-1} \tag{5}$$

where we have again used the density of states $g(\omega)$ to convert the sum over states to a frequency integral.

(d) Since $\Sigma = -PV$ we have

$$PV = -\Sigma = -k_BT\frac{V}{\pi^2}\frac{1}{c^3}\int_0^\infty d\omega\ \omega^2\ln(1-e^{-\beta\hbar\omega}) \tag{6}$$

Performing an integration by parts on the ω integral then yields

$$\begin{aligned}
\int_0^\infty d\omega\omega^2\ln(1-e^{-\beta\hbar\omega}) &= \int_0^\infty d\omega\frac{1}{3}\left(\frac{d}{d\omega}\omega^3\right)\ln(1-e^{-\beta\hbar\omega}) && (7)\\
&= \frac{1}{3}\omega^3\ln(1-e^{-\beta\hbar\omega})\Big|_0^\infty - \int_0^\infty d\omega\frac{1}{3}\omega^3\frac{d}{d\omega}\ln(1-e^{-\beta\hbar\omega}) && (8)\\
&= -\frac{1}{3}\int_0^\infty \omega^3\frac{\beta\hbar e^{\beta}\hbar\omega}{1-e^{\beta\hbar\omega}} && (9)\\
&= -\beta\hbar\frac{1}{3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1} && (10)
\end{aligned}$$

Thus we find that

$$PV = \frac{1}{3}\frac{V}{\pi^2}\frac{\hbar}{c^3}\int_0^\infty d\omega\frac{\omega^3}{e^{\beta\hbar\omega}-1} = \frac{1}{3}E \tag{11}$$

Problem 2

(a) It is convenient to first compute the Fermi wave vector k_F. If there are N electrons (each with 2 spin states so that $g_s = 2$) we have at $T = 0$

$$N = 2V\int\frac{d^3k}{(2\pi)^3}n(\mathcal{E}(\vec{k})) = V\int\frac{d^3k}{(2\pi)^3}\Theta(k_F - |\vec{k}|) = 2\frac{V}{(2\pi)^3}\frac{4}{3}\pi k_F^3 = V\frac{k_F^3}{3\pi^2} \tag{12}$$

Here we have used the fact that at $T = 0$ states for which $n(\mathcal{E}(\vec{k})) = 1$ for $|\vec{k}| < k_F$ and 0 for $|\vec{k}| > k_F$.

Thus the Fermi momentum is given by

$$p_F = \hbar k_F = \hbar \left(3\pi^2 \frac{N}{V} \right)^{1/3} \tag{13}$$

Note that this result does not depend in any way on the dispersion of the particles.

Finally, the Fermi energy is

$$\mathcal{E}_F = \hbar c k_F = \hbar c \left(3\pi^2 \frac{N}{V} \right)^{1/3} \tag{14}$$

(b) The total energy of this gas at $T = 0$ is given by

$$E = 2V \int \frac{d^3k}{(2\pi)^3} \hbar c |\vec{k}| \Theta(k_F - |\vec{k}|) = 2 \frac{V}{(2\pi)^3} \hbar c 4\pi \int_0^{k_F} k^3 dk = V \frac{\hbar c}{pi^2} \frac{1}{4} k_F^4 = V \frac{\hbar c}{4\pi^2} \left(3\pi^2 \frac{N}{V} \right)^{4/3} \tag{15}$$

(c) Using the fact that at $T = 0$ the pressure is given by $P = -(\partial E/\partial V)_N$ we find that

$$P = -\left(\frac{\partial E}{\partial V} \right)_N = \frac{1}{3} \frac{\hbar c}{4\pi^2} \left(3\pi^2 \frac{N}{V} \right)^{4/3} = \frac{1}{3} \frac{E}{V} \tag{16}$$

and so

$$PV = \frac{1}{3} E \tag{17}$$

(d) The grand partition function for this gas is given by

$$\mathcal{Z}(V, T, z) = \prod_{\vec{k},s} (1 + z e^{-\beta \hbar c |\vec{k}|}) \tag{18}$$

and so the grand potential is

$$\Sigma = -k_B T \ln \mathcal{Z} = -k_B T \sum_{\vec{k},s} \ln(1 + z e^{-\beta \hbar c |\vec{k}|}) = -k_B T \int d\mathcal{E} a(\mathcal{E}) \ln(1 + z e^{-\beta \mathcal{E}}) \tag{19}$$

where

$$a(\mathcal{E}) = 2 \int \frac{d^3k}{(2\pi)^3} \delta(\mathcal{E} - \hbar c |\vec{k}|) = \frac{V}{\pi^2} \frac{\mathcal{E}^2}{\hbar^3 c^3} \tag{20}$$

is the density of states for spin-1/2 (i.e. $g_S = 2$) ultrarelativistic particles.

Thus we obtain

$$\Sigma = -k_B T \frac{V}{\pi^2} \frac{1}{\hbar^3 c^3} \int_0^\infty d\mathcal{E} \mathcal{E}^2 \ln(1 + z e^{-\beta \mathcal{E}}) \tag{21}$$

(e) The total energy of this gas is given by

$$E = \sum_{\vec{k},s} \hbar c |\vec{k}| \frac{1}{z^{-1} e^{\beta \hbar c |\vec{k}|} - 1} = \int_0^\infty d\mathcal{E} a(\mathcal{E}) \frac{\mathcal{E}}{z^{-1} e^{\beta \mathcal{E}} - 1} = \frac{V}{\pi^2} \frac{1}{\hbar^3 c^3} \int_0^\infty d\mathcal{E} \frac{\mathcal{E}^3}{z^{-1} e^{\beta \mathcal{E}} - 1} \tag{22}$$

(f) Using the fact that $\Sigma = -PV$ we have that

$$\begin{aligned} PV &= -\Sigma = k_B T \frac{V}{\pi^2} \frac{1}{\hbar^3 c^3} \int_0^\infty d\mathcal{E} \frac{1}{3} \left(\frac{d}{d\mathcal{E}} \mathcal{E}^3 \right) \ln(1 + z e^{-\beta \mathcal{E}}) && (23) \\ &= \frac{1}{3} \frac{V}{\pi^2} \frac{1}{\hbar^3 c^3} \int_0^\infty d\mathcal{E} \frac{\mathcal{E}^3}{z^{-1} e^{\beta \mathcal{E}} - 1} && (24) \\ &= \frac{1}{3} E && (25) \end{aligned}$$

where the second line follows from an integration by parts (similar to that used in Part (d) of the previous problem).

Problem 3.

(a) For a two dimensional gas of spin-1/2 ($g_s = 2$) fermions we determine the Fermi wave vector k_F as follows

$$N = \sum_{\vec{k},s} n(\mathcal{E}(\vec{k})) = 2A \int \frac{d^2k}{(2\pi)^2} n(\mathcal{E}(\vec{k})) = 2A \frac{1}{(2\pi)^2} 2\pi \int_0^{k_F} k dk = \frac{A}{(2\pi)^2} \pi k_F^2 = A \frac{k_F^2}{2\pi} \tag{26}$$

Thus

$$k_F = \left(2\pi \frac{N}{A} \right)^{1/2} = (2\pi n)^{1/2} \tag{27}$$

where $n = N/A$ is the number density of this two-dimensional gas. The Fermi energy is then

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} 2\pi n \tag{28}$$

and the Fermi temperature is

$$T_F = \frac{E_F}{k_B} = \frac{\hbar^2}{2mk_B} 2\pi n \tag{29}$$

(b) The one-particle density of states for nonrelativistic spin-1/2 particles in two dimensions is

$$a(\mathcal{E}) = 2A \int \frac{d^2k}{(2\pi)^2} \delta(\mathcal{E} - \hbar^2 k^2/(2m)) = 2 \frac{A}{(2\pi)^2} 2\pi \int_0^\infty k dk \delta(\mathcal{E} - \hbar^2 k^2/(2m)) = A \frac{m}{\pi \hbar^2}, \quad \mathcal{E} > 0. \tag{30}$$

($a(\mathcal{E})$ vanishes for $\mathcal{E} < 0$.)

(c) At any temperature T, the number density of particles in this gas is given by

$$\frac{N}{A} = \int_0^\infty d\mathcal{E} \frac{a(\mathcal{E})}{A} \frac{1}{z^{-1} e^{\beta \mathcal{E}} + 1} = \frac{m}{\pi \hbar^2} \int_0^\infty d\mathcal{E} \frac{1}{z^{-1} e^{\beta \mathcal{E}} + 1} \tag{31}$$

Making the change of variables $y = \beta \mathcal{E}$ we find

$$n = \frac{N}{A} = \frac{m}{\pi \hbar^2} k_B T \int_0^\infty dy \frac{1}{z^{-1} e^y + 1} = 2 \left(\frac{2\pi m k_B T}{h^2} \right) \int_0^\infty dy \frac{1}{z^{-1} e^y + 1} \tag{32}$$

The integral can easily be done analytically with the result

$$\int_0^\infty dy \frac{1}{z^{-1} e^y + 1} = \ln(1 + z) \tag{33}$$

Thus we find

$$n = 2 \left(\frac{2\pi m k_B T}{h^2} \right) \ln(1 + z) \tag{34}$$

Solving for the fugacity then yields

$$z = e^{nh^2/(4\pi m k_B T)} - 1 = e^{n\pi \hbar^2/(m k_B T)} - 1 = e^{E_F/(k_B T)} - 1 \tag{35}$$

and, using the fact that

$$z = e^{\mu/k_B T}, \tag{36}$$

we find the following expression for μ

$$\begin{aligned} \mu &= k_B T \ln z = k_B T \ln \left(e^{E_F/(k_B T)} - 1 \right) = k_B T \ln \left(e^{E_F/(k_B T)} \left(1 - e^{-E_F/(k_B T)} \right) \right) && (37) \\ &= k_B T \left(\ln e^{E_F/(k_B T)} + \ln \left(1 - e^{-E_F/(k_B T)} \right) \right) && (38) \\ &= E_F + k_B T \ln \left(1 - e^{-E_F/(k_B T)} \right) && (39) \end{aligned}$$

Finally, from Eq. (35) we also see that when $T \gg T_F$

$$\begin{aligned} z &\simeq 1 + \frac{T_F}{T} + \cdots - 1 \simeq \frac{T_F}{T} && (40) \\ &= \frac{\hbar^2}{2mk_BT} 2\pi \frac{N}{A} && (41) \\ &= \frac{1}{2} \frac{h^2}{2\pi m k_B T} \frac{N}{A} && (42) \\ &= \frac{1}{2} \frac{\lambda^2}{l^2} && (43) \end{aligned}$$

where $l = (A/N)^{1/2}$ is the mean particle spacing and $\lambda = (h^2/(2\pi m k_B T))^{1/2}$ is the thermal wave length.