PHZ3113 Mathematical Physics
Elementary Fourier series

Rather than expanding a function in a power series, $\sum_n a_n x^n$, it can often be useful to expand in a series of sines and/or cosines. This is especially true if the function is periodic or discontinuous. Since a Fourier series is not absolutely convergent, its infinite sum could be discontinuous, even though the individual terms are continuous.

The expansion of a function $f(x)$ that is periodic in x with period $2L$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \ ,$$

where

$$a_n = \frac{1}{L}\int_{-L}^{+L} f(x)\cos(n\pi x/L)\mathrm{d}x$$

and

$$b_n = \frac{1}{L}\int_{-L}^{+L} f(x)\sin(n\pi x/L)\mathrm{d}x$$

for $n = 0,\ 1,\ 2,\ \ldots$. ($b_0 = 0$ and a_0 is twice the average of $f(x)$ over its period).

Example 1:

Sawtooth wave.

$$f(x) = x \quad \text{for } -L < x < +L$$

and $f(x + 2L) = f(x)$.

Here, since this is an odd function, $a_n = 0$ by symmetry and

$$b_n = \frac{1}{L}\int_{-L}^{+L} x\sin(n\pi x/L)\mathrm{d}x = (-1)^{n+1}\frac{2L}{n\pi} \ ,$$

where the integration involved integration by parts. See figure on the website. As the number of terms increases, the finite series approximates the discontinuities of this function better and better.

Example 2:

Triangular wave.

$$f(x) = |x| \quad \text{for } -L \le x \le +L$$

and $f(x + 2L) = f(x)$.

Here, since this is an even function, $b_n = 0$ by symmetry, $a_0 = L$, and

$$a_n = \frac{1}{L}\int_{-L}^{+L} |x|\cos(n\pi x/L)\mathrm{d}x = \frac{2L}{(n\pi)^2}\left[(-1)^n - 1\right] \quad \text{for } n \ge 1.$$

The integration again involved integration by parts. See figure on the website.